Optimal Multivariate Financial Decision Making*

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Abstract

Agents who pursue optimal portfolio choice by optimizing a univariate objective (e.g., an expected utility) obtain optimal payoffs that are increasing with each other (comonotonic). This situation may lead to an undesirable level of systemic risk for society. A regulator may therefore aim to enforce diversification among the various portfolios by optimizing a suitable multivariate objective. We assess the cost of diversification and provide the strategy that the regulator should pursue for obtaining the desired level of diversification.

Keywords: Portfolio Optimization; Cost-Efficiency; Multivariate Preferences; Diversification; Systemic Risk.

1 Introduction

Our work is motivated by the observation that when agents are maximizing their expected utility of terminal wealth - in isolation from each other - their optimal allocations are all long with the market and thus comonotonic with each other (i.e., they move up and down simultaneously). This represents a true societal problem in that economic hardship hits all agents simultaneously. The reason for this observation is that the optimal decision for each agent depends solely on the probability distribution of the payoff (given the law-invariance of expected utility theory) and of its cost (due to the increasingness of the utility function). Clearly, optimal payoffs are those that achieve some distribution function (depending on the utility function at hand) at cheapest possible cost, i.e., are "cost-efficient".[1] It is then easy to see that all cost-efficient payoffs are increasing with the market asset and thus with each other. In fact, this reasoning is not restricted to agents with expected utility preferences, but applies to all agents with increasing and law

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[1] This notion of cost-efficiency dates back to the works of Dybvig (1988a,b) and Bernard et al. (2014), which focus on finding the cheapest payoff that has a given probability distribution of wealth.

invariant preferences, or, equivalently, agents who have preferences consistent with first-order stochastic dominance (Bernard et al. 2015, Theorem 1).[2]

In this paper, we take the viewpoint of a regulator or a social planner who is responsible for the supervision of d agents acting in the same market (such as d financial institutions). If each of these d agents has preferences consistent with first-order stochastic dominance, then their optimal decisions are all comonotonic. In such a situation, the risk that all agents incur big losses simultaneously is maximum, as none of the strategies provides a hedge against the others. Thus, if these agents were portfolio managers of large financial institutions, no diversification would take place and the economy would be exposed to high systemic risk. In order to protect the system from collapse due to simultaneous defaults, the regulator may step in and impose a desired level of diversification among the portfolios. Specifically, the regulator may quantify the global cost, or additional capital, needed to keep the same distribution of returns for each of the banks while ensuring that they show the desired dependence structure. We name this extra capital the "cost of diversification". This collected money can then be used to protect the system.

A natural objective function to optimize for the regulator is the minimization of a multivariate risk measure (e.g., based on a systemic risk measure) or the maximization of a multivariate expected utility $U(X_1, \ldots, X_d)$ in which each X_i corresponds to the value of the portfolio of assets of each institution net of liabilities. Unfortunately, there is no unique method to solve such multivariate optimal portfolio choice. Here, we propose an alternative approach by characterizing optimal multivariate allocations. Based on the observation that projects having the same joint distribution also have the same expected utility, a rational investor who worries only about the expected utility and has increasing preferences will merely be interested in finding the cheapest (cost-efficient) way for $\left(X_1, \ldots, X_d\right)$ to achieve this joint distribution (multivariate cost-efficiency). This reasoning applies to very general settings that are law invariant and increasing in at least one of the components (and beyond the expected utility theory). In this paper, we characterize multivariate cost-efficient payoffs and provide an explicit algorithm to determine them. Under some specific assumptions, we obtain the optimal payoff explicitly and we use this result to verify the accuracy of the algorithm we propose. We then show how this construction can be used by a regulator to control for systemic risk.

Our work builds on several strands of literature. In addition to the aforementioned results on cost-efficiency, central to our study is the concept of multivariate utility maximization, initially introduced by Richard (1975) and then applied, for instance, in Deelstra et al. (2001), Kamizono (2004), Bouchard and Pham (2005) and Campi and Owen (2011) to optimal consumption and portfolio problems. More generally, we contribute to the literature on continuous-time portfolio selection problems in complete markets with multivariate preferences, a field extending the seminal contribution of Merton (1969, 1971) that focused on univariate preferences.[3]

For the desired properties of multivariate preferences, we refer to the extensive literature on decision analysis (among others, Eeckhoudt and Schlesinger (2006), Eeckhoudt et al. (2007), Eeckhoudt et al. (2009),

[2] This includes a large set of models for optimal portfolio selection, e.g., Yaari's duality theory (Yaari (1987)), cumulative prospect theory (Tversky and Kahneman (1992), quantile-, VaR- and CVaR-based models (Alexander et al. (2006), Cherny and Madan (2009), He and Zhou (2011), He et al. (2015), Al Janabi et al. (2017), Ahmadi-Javid and Fallah-Tafti (2019)) and others (Browne (2000), Bernard et al. (2019b)).

[3] The approach pursued by this stream of literature is to optimize over sets of continuously rebalanced investment strategy. Clearly, a prominent alternative is the "static" framework developed by Markowitz (1952) and ensuing works, which consists in finding optimal, constant portfolio weights to be invested in a number of assets for the entire time horizon.

Tsetlin and Winkler (2009) Crainich et al. (2013)). In the context of risk-transfer and risk-sharing, results related to ours with respect to diversified and/or comonotonic allocations can be found in Chateauneuf et al. (2000), Filipović and Kupper (2008), Jouini et al. (2008), Ludkovski and Rüschendorf (2008), Carlier et al. (2012), Schumacher (2018), Bernard et al. (2018) and Liebrich and Svindland (2019). Finally, we refer to Pazdera et al. (2016) and Alserda et al. (2019) for analysis of cooperative-collective investment with heterogeneous preferences, as well as to the more recent paper by Chen et al. (2021) for an investigation on optimal collective investment in the presence of portfolio insurance.

The rest of the paper is organized as follows. In Section 2 we introduce the problem of multivariate portfolio allocation. Section 3 provides the notion of multivariate cost-efficiency that is key to the characterization of the solution of a law invariant multivariate maximization problem. In Section 4 we propose an explicit procedure to construct optimal (cost-efficient) multidimensional payoffs, i.e., we construct the cheapest multidimensional payoff that achieves a given multivariate distribution. We verify the accuracy of the method in the situation of a multivariate Gaussian distribution for which we show that optimal payoffs can be found explicitly. In Section 5 we illustrate our results with estimation of the cost of diversification in several practical situations and provide a concrete example of how it can be used in practice to reduce systemic risk. We conclude in Section 6. Longer proofs are relegated to the appendix.

2 Setting

We assume a complete, frictionless financial market $(\Omega, \mathcal{F}, \mathbb{P})$ with a fixed investment horizon $T > 0$. The market has a unique pricing kernel ξ_T, which is a positive, integrable random variable on $\mathbb{R}_+\backslash\{0\}$, so that the value X_0 at time 0 of a payoff X_T at time T is computed as $X_0 = \mathbb{E}\left[\xi_T X_T\right]$. We consider only terminal payoffs X_T such that X_0 is finite. $X_T \geqslant 0$ refer to gains (income) and $X_T \leqslant 0$ refer to losses.

One of the main results that we aim to extend is the concept of *cost-efficiency* (Dybvig (1988a,b)), which is conveniently recalled here.

Lemma 2.1 (Cost-efficiency)**.** *Let ξ_T be continuously distributed on $\mathbb{R}^+\backslash\{0\}$, with CDF F_{ξ_T}. The cheapest (cost-efficient) way to achieve a final portfolio payoff X_T with distribution F at time horizon T, that is, the solution to the problem*

$$\min_{X_T \sim F} \mathbb{E}\left[\xi_T X_T\right], \tag{1}$$

is almost surely (a.s.) unique and given by $X_T^* = F^{-1}\big(1 - F_{\xi_T}(\xi_T)\big)$ a.s., which is non-increasing in ξ_T.

The proof of Lemma 2.1 is given in full detail in the literature; see, e.g., Theorem 1 in Dybvig (1988a) for the case of a discrete market or Corollary 1 of Bernard et al. (2014) for the general case. As will become clearer afterwards, this result will be key to prove the multivariate extension discussed in Section 3.

2.1 Motivation

Consider a regulator observing the portfolios of d institutions and aiming at quantifying the global risk that is implied by the joint distribution of these portfolios. For example, Doldi and Frittelli (2019, 2021) propose the following objective function in the context of systemic risk and risk transfer equilibrium:

$$U(x_1, \dots, x_d) = \sum_{i=1}^{d} U_i(x_i) + \Lambda(x_1, \dots, x_d), \tag{2}$$

where $U_i, i = 1, \dots, d$, are univariate utility functions and $\Lambda : \mathbb{R}^d \to \mathbb{R}$ is defined as a concave *aggregator* that, say, could be imposed on the agents by a regulator. A candidate class of aggregators proposed by the authors is of the form

$$\Lambda(x_1, \dots, x_d) = u\left(\sum_{i=1}^{d} \beta_i x_i\right), \quad \beta_i \geqslant 0, \tag{3}$$

where $u : \mathbb{R} \to \mathbb{R}$ is concave.

It can easily be shown that multivariate utility functions as in (3) are submodular utilities.[4] Throughout the literature on multivariate preferences, submodularity appears as a desirable property (Eeckhoudt and Schlesinger (2006), Eeckhoudt et al. (2007), Eeckhoudt et al. (2009), Crainich et al. (2013)). This property is in fact closely related to the concept of *multivariate risk aversion* (Richard (1975)) or *correlation aversion* (in the terminology of Epstein and Tanny (1980)), according to which, simply put, individuals "prefer a 50-50 gamble of a loss in wealth or a loss in health over another 50-50 gamble offering a loss in neither dimension or a loss in both" (Eeckhoudt et al. (2007)). In the context of portfolio selection, submodular utility functions lead then to less extreme outcomes *across* portfolios (multivariate risk aversion), an attribute that naturally extends the basic assumption that individuals prefer allocations with less extreme outcomes (univariate risk aversion). In an intertemporal framework, Andersen et al. (2018) provide empirical support to this claim.

We could then aim at searching for the optimal situation - from the viewpoint of the regulator - for some submodular utility function U:

$$\max_{(X_1,\dots,X_d)} \mathbb{E}\left[U\big(X_1, \dots, X_d\big)\right], \qquad \text{s.t. } \mathbb{E}\left[\xi_T \sum_{i=1}^{d} X_i\right] \leqslant w_0, \tag{4}$$

where w_0 denotes a given budget.[5] However, to the best of our knowledge, there are no explicit solutions in the literature in the case of submodular multivariate utility functions.

In fact, such an expected utility problem (4) has only been solved explicitly in cases that lead to comonotonic allocations (i.e., all X_i are increasing with each other). For instance, multivariate utility functions that have been studied extensively throughout the literature are the additive multivariate utility, i.e., $U(x_1, \dots, x_d) = \sum_{i=1}^{d} U_i(x_i)$, where each U_i is a univariate concave utility function, or the Cobb-Douglas utility (Cobb and Douglas (1928), Campi and Owen (2011)), which is defined on $\mathbb{R}_+^d$ by $U_\beta(x_1, \dots, x_d) = \prod_{i=1}^{d} x^{\beta_i}, d \geqslant 2$, such that $\beta_i > 0$ for all i, $\sum_{i=1}^{d} \beta_i < 1$.[6]

[4] A function $g : \mathbb{R}^d \to \mathbb{R}$ is submodular if $g(\vec{x} \wedge \vec{y}) + g(\vec{x} \vee \vec{y}) \leqslant g(\vec{x}) + g(\vec{y})$, for all $\vec{x}, \vec{y} \in \mathbb{R}^d$, where $\wedge(\vee)$ denotes the componentwise minimum (maximum) of $\vec{x}$ and $\vec{y}$.

[5] We assume a global budget constraint, whereas often in the literature it has been discussed how to handle individual budget constraints. Also, throughout the paper we assume that every $X_i, i = 1, \dots, d$, share the same investment horizon T, so we omit any time-related notation.

[6] More generally, one can consider utility functions of the form $U(x_1, \dots, x_d) = \prod_{i=1}^{d} U_i(x_i)$ for $U_i, i = 1, \dots, d$, positive and concave.

Additive utilities and Cobb-Douglas share the property of being supermodular.[7] The following theorem characterizes the solution of a maximum expected utility problem for a supermodular utility function.

Theorem 2.2. *Consider a supermodular function $U : \mathbb{R}^d \mapsto \mathbb{R}$, which is strictly increasing in at least one of the components x_{j_0}, for $j_0 \in \{1, \dots, d\}$. Assume that the maximum expected utility problem*

$$\max_{(X_1,\dots,X_d)} \mathbb{E}\left[U\big(X_1,\dots,X_d\big)\right], \qquad s.t.\ \mathbb{E}\left[\xi_T \sum_{i=1}^d X_i\right] \leqslant w_0, \tag{5}$$

has a solution $(X_1^*, \dots, X_d^*)$ with marginal distributions F_i^*, $i = 1, \dots, d$. Then, this solution must be of the form

$$X_i^* = f_i(\xi_T)\ a.s., \quad i = 1, \dots, d,$$

for some decreasing function f_i. Furthermore, $f_i(\cdot) = (F_i^*)^{-1}(1 - F_{\xi_T}(\cdot))$.

The proof of Theorem 2.2 relies on Lemma 2.1 and is relegated to Appendix A. From Theorem 2.2, we can infer that the optimization of a d-dimensional supermodular multivariate utility can always be reduced to d individual (univariate) expected utility optimizations. The following result establishes this claim formally.

Theorem 2.3. *Consider a supermodular utility function $U : \mathbb{R}^d \mapsto \mathbb{R}$ which is strictly increasing in at least one of the components x_{j_0}, for $j_0 \in \{1, \dots, d\}$. When it exists, the optimal choice $(X_1^*, \dots, X_d^*)$ that maximizes (5) with a budget constraint w_0 also maximizes d individual optimizations. Specifically, X_i^* for all $i \in \{1, \dots, d\}$ solves*

$$\max_{X_i} \mathbb{E}\left[U_i(X_i)\right], \qquad s.t.\ \mathbb{E}\left[\xi_T X_i\right] = w_i, \tag{6}$$

where $U_i(x) = \int_c^x F_{\xi_T}^{-1}(1 - F_i^*(y))dy$, with F_i^* denoting the CDF of X_i^*, c is chosen such that $F_i^*(c) > 0$, and $w_i = \mathbb{E}\left[\xi_T X_i^*\right] = \mathbb{E}\left[\xi_T \left(F_i^*\right)^{-1}\left(1 - F_{\xi_T}(\xi_T)\right)\right]$.

Proof. Define $w_i = \mathbb{E}\left[\xi_T X_i^*\right]$ and denote by F_i^* the CDF of X_i^*. Using Theorem 2 of Bernard et al. (2015), X_i^* is the optimal solution to the expected utility maximization (6) with the generalized concave utility function U_i as defined in the statement of the theorem. □

Remark 2.4. We emphasize that Theorem 2.3 does not entail that the solution of the (supermodular) collective and individual optimization problems generally coincide, but rather that the two solutions coincide for a specific choice of the budget w_i allocated to each of the individual optimal portfolios. For an illustration of this result in the case of a power utility, see Appendix B.

In summary, in the case of a supermodular utility function optimal payoffs exhibit a comonotonic dependence, but unfortunately supermodularity is generally not seen as a desirable property of utility functions. By contrast, while submodularity is accepted as a desirable property of multivariate utility

[7] A function $g : \mathbb{R}^d \to \mathbb{R}$ is supermodular if $g(\vec{x} \wedge \vec{y}) + g(\vec{x} \vee \vec{y}) \geqslant g(\vec{x}) + g(\vec{y})$, for all $\vec{x}, \vec{y} \in \mathbb{R}^d$. Note that a twice differentiable multivariate utility function $U : \mathbb{R}^d \to \mathbb{R}$ is supermodular if and only if $\frac{\partial^2 U(x_1,\dots,x_d)}{\partial x_j \partial x_k} \geqslant 0$ for all $k \neq j$. Thus, if $U(x_1, \dots, x_d) = \prod_{i=1}^d U_i(x_i)$, it is easy to see that $\frac{\partial^2 U(x_1,\dots,x_d)}{\partial x_k \partial x_j} = U_j'(x_j)U_k'(x_k)\prod_{\substack{i=1 \\ i \neq k,j}}^d U_i(x_i) \geqslant 0, k \neq j$. For results on (continuous-time) portfolio selection problems under utility functions of the form $U(x, y) = g(x)h(y)$, we refer, among others, to Zariphopoulou (2001), Tehranchi (2004) and Henderson (2005).

functions, it appears very difficult to determine optimal payoffs when utility functions are submodular. In the next section, nonetheless, we provide a characterization of optimality of multivariate payoffs. We emphasize that such approach to finding optimal payoffs is of interest in its own right: rather than specifying a multivariate utility function upfront and deriving for a given budget a payoff with maximum expected utility, an agent may wish to specify a joint distribution G and aiming to obtain a payoff distributed with G at minimum possible cost. Here we also refer to Section 5 where we apply these results to the case of a regulator who wishes to manage systemic risk.

3 Multivariate Cost-Efficiency

In this section we introduce a multivariate extension of the problem in (1). We also provide one of the key results of this paper, Theorem 3.2, which characterizes the solution of any law invariant multivariate utility maximization problem in terms of cost-efficiency. Formally, the cost-efficiency problem can be formulated in the d-dimensional case as follows:

$$\min_{(X_1,\dots,X_d)\sim G} \mathbb{E}\left[\xi_T \sum_{i=1}^{d} X_i\right], \tag{7}$$

where G is a d-dimensional distribution. Solutions to problem (7) are called multi-dimensional cost-efficient. In the following section we first provide an approach by which to approximate the solution to this problem under very general conditions and second, we solve (7) explicitly in the special case where G is a multivariate Gaussian distribution; see Section 4.2.

Building on existing literature on cost-efficiency, the solution to the multivariate cost-efficiency problem (7) satisfies the following theorem, which is the key result needed to build the discrete procedure that makes it possible to compute an approximate solution (see Section 4.1).

Theorem 3.1. *Assume that an optimal solution $(X_1,\dots,X_d)$ to (7) exists. Then $cov\left(\sum_{i=1}^d X_i, \xi_T\right)$ is minimum, and thus $\sum_{i=1}^d X_i$ is anti-monotonic with ξ_T. Furthermore, when ξ_T is continuously distributed, the optimal solution $(X_1^*,\dots,X_d^*)$ must satisfy*

$$\sum_{i=1}^{d} X_i^* = H^{-1}\left(1 - F_\xi(\xi_T)\right),$$

where H denotes the probability distribution of $\sum_{i=1}^d X_i$ for $(X_1,\dots,X_d)\sim G$.

The proof is given in Appendix C.

Consider general multivariate preferences V and an agent aiming to solve

$$\max_{(X_1,\dots,X_d)} V(X_1,\dots,X_d), \qquad \text{s.t. } \mathbb{E}\left[\xi_T \sum_{i=1}^{d} X_i\right] = w_0. \tag{8}$$

Endowed with some initial budget w_0, the agent is then handling several projects $X_1,\dots,X_d$ following a joint distribution G and is achieving satisfaction $V(X_1,\dots,X_d)$ from them. For instance, for $V(X_1,\dots,X_d) := \mathbb{E}\left[U(X_1,\dots,X_d)\right]$, this setting covers the maximization of a multivariate expected utility. Alternatively, one can also look at the problem as the minimization of a multivariate risk measure ρ, thus setting $V\big(X_1,\ldots,X_d\big) := -\rho(X_1,\ldots,X_d)$.

Observe that two d-dimensional allocations having the same joint distribution also have the same expected multivariate utility. A rational investor who cares only about the expected utility and has increasing preferences will only be interested in finding the cheapest portfolio with this given joint distribution (i.e., the multivariate cost-efficient strategy solving problem (7)). Assuming for simplicity that V is increasing in each component, it is then clear that the multivariate cost-efficient strategy $\big(X_1^*,\ldots,X_d^*\big)$ solving (7) for the joint distribution G makes it possible to achieve the same level of expected utility (by also having the joint distribution G) at a (strictly) lower cost. This reasoning can be formalized more generally to multivariate utility functions that are increasing in only one of the components and extended beyond the expected utility setting to more general preferences, as clarified in the following result.

Theorem 3.2. *Consider an investor with law invariant preferences and who is maximizing her objective function $V\big(X_1,\ldots,X_d\big)$ with a given initial budget w_0, i.e., $\mathbb{E}\left[\xi_T\sum_{i=1}^d X_i\right] = w_0$ (Problem 8). Also, assume that $V(\cdot)$ is strictly increasing in at least one of the d components. Then the optimal investment for this investor, when it exists, is multivariate cost-efficient, i.e., it solves (7) for some joint distribution G.*

Proof. Consider the general problem (8). Denote by G the joint distribution of an optimal solution, when it exists. It is clear that the multivariate cost-efficient strategy $\big(X_1^*,\ldots,X_d^*\big)$ solving (7) is such that it also has the joint distribution G, and thus it achieves the same level of preference due to the law-invariance property of V. In other words, $V\big(X_1,\ldots,X_d\big) = V\big(X_1^*,\ldots,X_d^*\big)$.

The last claim can be proved by contradiction. Assume that the optimal solution of (8) is not cost-efficient. Then the budget $w_0^* := \mathbb{E}\left[\xi_T\sum_{i=1}^d X_i^*\right]$ needed to generate $\big(X_1^*,\ldots,X_d^*\big)$ is strictly smaller than w_0. Let x_{j_0} be a component such that $V\big(x_1,\ldots,x_{j_0},\ldots,x_d\big)$ is increasing in it. We find that $V\big(X_1^*,\ldots,X_{j_0}^*+(w_0-w_0^*)e^{rT},\ldots,X_d^*\big) > V\big(X_1^*,\ldots,X_{j_0}^*,\ldots,X_d^*\big) = V\big(X_1,\ldots,X_d\big)$ and that the multivariate payoff $(X_1^*,\ldots,X_{j_0}^*+(w_0-w_0^*)e^{rT},\ldots,X_d^*)$ requires exactly a budget w_0. This last statement contradicts the optimality of $\big(X_1,\ldots,X_d\big)$ to (8). Thus the optimal solution of (8) must be multivariate cost-efficient (so that $w_0 = w_0^*$). ☐

Theorem 3.2 can then be directly used to improve a multivariate allocation, each time this allocation is not multivariate cost-efficient, i.e., does not solve the original problem (7). This result, together with Theorem 3.1, provides a characterization of an optimal multivariate allocation maximizing a multivariate increasing law invariant objective, as we formalize in the following corollary.

Corollary 3.3. *When it exists, an optimal solution $(X_1^*,\ldots,X_d^*)$ to Problem (8), for some law invariant $V(X_1,\ldots,X_d)$ that is increasing in at least one component x_{j_0}, must satisfy that $\sum_{i=1}^d X_i^*$ is antimonotonic with ξ_T.*

4 Deriving Multivariate Cost-Efficient Portfolios

In this section, we first present our new discrete procedure to solve the multivariate cost-efficiency problem introduced in (7) for a given general distribution G. Then, in the special case when G is a multivariate Gaussian distribution and the financial market is lognormal, we solve the problem explicitly and provide a closed-form expression of the cheapest multivariate payoff that achieves a given joint distribution. The last section discusses how this payoff can be replicated in a lognormal market using a path-dependent claim.

4.1 Discrete Approach for Solving Problem (7)

The multivariate cost-efficiency problem (7) can be solved via a discretization approach that is directly inspired by Theorem 3.1 and Corollary 3.3. In particular, our procedure relies on the approximation of a continuously distributed pricing kernel ξ_T by a discretized distribution taking n distinct equiprobable values.

Algorithm 4.1. Consider the following steps:

1. Build an $n \times (d+1)$-dimensional matrix:
   - In the first column, discretize the state-price process ξ_T and report the n discretized values (assuming, for instance, without loss of generality, that $\xi_{T,1} < \xi_{T,2} < \cdots < \xi_{T,n}$).
   - In the other d columns, each of the n rows contains an independent realization of the joint d-dimensional distribution of $(X_1, \ldots, X_d)$:

$$\left(\begin{array}{c|cccc} \xi_{T,1} & x_{11} & x_{21} & \ldots & x_{d1} \\ \xi_{T,2} & x_{12} & \ddots & \ddots & x_{d2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \xi_{T,n} & x_{1n} & x_{2n} & \ldots & x_{dn} \end{array}\right). \tag{9}$$

2. Compute $S_i := \sum_{j=1}^{d} x_{i,j}$ for each realization $i = 1, \ldots, n$.

3. Order the block of columns "$X_1, \ldots, X_d$" such that the row sums S_i across these columns are oppositely ordered to the values appearing in the first column $\xi_{T,i}$ for each row $i = 1, \ldots, n$.

We collect this result in the proposition below.

Proposition 4.2. *Applying Algorithm 4.1 leads to an approximation of the optimal solution to the multivariate cost-efficiency problem (7). Namely, the optimal solution $(X_1^*, \ldots, X_d^*)$ is given by the last d columns of the output matrix (9).*

The n realizations in the numerical approach are typically obtained via Monte Carlo simulation and represent a sample of n equiprobable realizations from the joint distribution of $\left(X_1, \ldots, X_d\right)$. The procedure to solve (7) is very simple and needs one single rearrangement of the matrix. This rearrangement achieves the minimum cost for the portfolio sum and preserves the joint distribution of $\left(X_1, \ldots, X_d\right)$. This fact is immediate as we have preserved the order inside the n rows in the last d columns of the matrix (doing a "block" rearrangement). The solution to (7) is then obtained by observing that this matrix provides an approximation of the joint distribution of $\left(\xi_T, X_1, \ldots, X_d\right)$, which can be also written as a degenerate distribution. Specifically, we can immediately find functions f_i^* such that the column i is a function of the first column. This is the solution to our problem:

$$X_i^* = f_i^*(\xi_T), \quad i = 1, \ldots, d. \tag{10}$$

This new vector $(X_1^*, \dots, X_d^*)$ improves on the original one in that it has the same joint distribution but a strictly lower cost unless it is identically equal to $\left(X_1, \dots, X_d\right)$.

Intuitively, the procedure above provides a reallocation of n outcomes of $\left(X_1, \dots, X_d\right)$ over the n equiprobable states (corresponding to the n rows of the matrix, and each row corresponds to a state that is characterized by the value of the state-price process).

Remark 4.3. The discrete procedure presented above solves the multivariate cost-efficiency problem (7) exactly for any multivariate discrete distribution G over a set of n equiprobable states. In this case, the matrix representation in (9) is indeed an exact description of the joint distribution of $(X_1, \dots, X_d)$ in which each state of the world corresponds to one row in the matrix. If the sum takes the same value in several rows then the solution is not unique. For a general joint distribution, one can always improve the accuracy of the approximation by discretizing further, and therefore such representation (9) can be made as accurate as desired. Thus, the d-dimensional strategy $\left(X_1, \dots, X_d\right)$ that minimizes the cost of achieving the joint multivariate distribution G is expressed as $X_i = f_i^*(\xi_T)$, in which f_i^* can be known more accurately by simulating a larger sample (increasing n), for instance, via a Monte Carlo method.

Remark 4.4. A natural situation in which we can apply the characterization given in Theorem 3.2 can be described as follows. Consider a portfolio manager with investments in d different markets (such as the foreign exchange market, commodities market, energy market, etc.). Applying the procedure described in Proposition 4.2, the manager can construct a multivariate cost-efficient portfolio that achieves the same joint distribution at a strictly lower cost if the allocation was not already cost-efficient.

Remark 4.5. In the special case of a multivariate Gaussian distribution, we are able to derive the solution to the multivariate cost-efficiency problem explicitly and thus to check the accuracy of the discrete procedure in this case (Section 4.2.1).

4.2 Special Case of a Multivariate Gaussian Distribution

In this section, we solve problem (7) explicitly when the target distribution is a multivariate Gaussian distribution and the market is lognormal. We then use this setting in Section 4.2.1 to verify the validity and accuracy of the numerical procedure presented above.

Let T denote the investment horizon. In a so-called lognormal market, the state-price process ξ_T is lognormally distributed with parameters μ_{ξ_T} and σ_{ξ_T}, i.e., $\xi_T \sim \log\mathcal{N}(\mu_{\xi_T}, \sigma_{\xi_T})$. For ease of presentation, we assume hereafter that the parameters describing the market are constant and that ξ_T can simply be written as

$$\xi_T = e^{-rT - \frac{\theta^2 T}{2} - \theta W_T}, \tag{11}$$

where $W_T \sim \mathcal{N}\left(0, \sqrt{T}\right)$. Then, μ_{ξ_T} and σ_{ξ_T} are given by $\mu_{\xi_T} = -rT - \frac{\theta^2 T}{2}$ and $\sigma_{\xi_T} = \theta\sqrt{T}$. A special case of the lognormal market in (11) is the Black-Scholes market. In the latter case we have $\theta = (\mu - r)/\sigma$, where μ is the drift and σ is the volatility of the underlying risky asset $S_T = S_0 e^{\left(\mu - \sigma^2/2\right)T + \sigma W_T}$. In fact, in this case it holds that

$$\xi_T = \alpha \left(\frac{S_T}{S_0}\right)^{-\beta}, \tag{12}$$

where $\alpha = \exp\left(-rT - \frac{\theta^2 T}{2} + \beta\left(\mu - \frac{\sigma^2}{2}\right)T\right)$ and $\beta = \theta/\sigma$.

Furthermore, assume that the investor wants to achieve a multivariate normal distribution $(X_1, \ldots, X_d)$ with correlation matrix $C = (\rho_{ij})_{1\leqslant i,j\leqslant d}$, vector of means $(\mu_i T)_{1\leqslant i\leqslant d}$ and standard deviations $(\sigma_i\sqrt{T})_{1\leqslant i\leqslant d}$. Each X_i, $i = 1, \ldots, d$, is normally distributed $\mathcal{N}\left(\mu_i T, \sigma_i\sqrt{T}\right)$.

In the case of a lognormal market with investment horizon T as in (11), and when the target multivariate distribution G of $\left(X_1, \ldots, X_d\right)$ in (7) is a multivariate Gaussian distribution, we can solve explicitly for the expressions $f_i^*, i = 1, \ldots, d$, in (10). We show this result in the following proposition.

Proposition 4.6. *Let G be a multivariate Gaussian distribution with vectors of means and standard deviations $\vec{\mu} = (\mu_1, \ldots, \mu_d)$ and $\vec{\sigma} = (\sigma_1, \ldots, \sigma_d)$, respectively, and correlation matrix $C = (\rho_{ij})_{1\leqslant i,j\leqslant d}$. The optimal solution $\left(X_1^*, \ldots, X_d^*\right)$ to problem (7) is such that $\left(X_1^*, \ldots, X_d^*, \ln(\xi_T)\right)$ follows a multivariate Gaussian distribution with correlation matrix*

$$\widetilde{C} = \left(\begin{array}{cccc|c} 1 & \rho_{12} & \cdots & \rho_{1d} & a_1 \\ \rho_{12} & \ddots & \ddots & \rho_{2d} & a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \rho_{d1} & \rho_{d2} & \cdots & 1 & a_d \\ \hline a_1 & a_2 & \cdots & a_d & 1 \end{array}\right), \quad \vec{a}^{\intercal} := (a_1, \ldots, a_d)^{\intercal} = -\frac{C\vec{\sigma}}{\sqrt{\vec{\sigma}^{\intercal}\, C\, \vec{\sigma}}}. \tag{13}$$

The correlation matrix $\widetilde{C}$ characterizes the dependence between each X_i^* and ξ_T, and thus ultimately the relationship in (10).[8]

The proof is reported in Appendix D.

Two special cases of Proposition 4.6 lead to considerable simplifications. In Corollary 4.7 we consider the case in which the correlation matrix is homogeneous. In Corollary 4.8 we give the explicit expressions in two and three dimensions. The proof of the corollaries is a straightforward application of Proposition 4.6 and the formula in (13), and is therefore omitted.

Corollary 4.7. *Homogeneous case: $d \geqslant 2$. Let $\vec{\mu} = (\bar{\mu}, \ldots, \bar{\mu}) \in \mathbb{R}^d$ and for all $i \neq j$ assume that $\rho_{i,j} = \rho > -\frac{1}{d-1}$. Thus, $\left(X_1^*, \ldots, X_d^*, \ln(\xi_T)\right)$ follows a multivariate Gaussian distribution with correlation matrix*

$$\widetilde{C} = \left(\begin{array}{cccc|c} 1 & \rho & \cdots & \rho & \bar{a} \\ \rho & \ddots & \ddots & \rho & \bar{a} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \rho & \rho & \cdots & 1 & \bar{a} \\ \hline \bar{a} & \bar{a} & \cdots & \bar{a} & 1 \end{array}\right), \quad \bar{a} := -\sqrt{\frac{1 + (d-1)\rho}{d}}.$$

Corollary 4.8. *Non-homogeneous case: $d = 2$ or $d = 3$.*

- *($d = 2$) Let $\vec{\mu} = (\mu_1, \mu_2), \vec{\sigma} = (\sigma_1, \sigma_2)$ and $C = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix}$. Thus, $\left(X_1^*, X_2^*, \ln(\xi_T)\right)$ follows a*

[8] Notably, under the setting of Proposition (4.6), the vector $\vec{a}$ in (13) does not depend on the parameters describing the pricing kernel ξ_T.

trivariate Gaussian distribution with correlation matrix

$$\widetilde{C} = \left(\begin{array}{cc|c} 1 & \rho_{12} & a_1 \\ \rho_{12} & 1 & a_2 \\ \hline a_1 & a_2 & 1 \end{array}\right), \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{\sigma_1 + \sigma_2\rho_{12}}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho_{12}}} \\ -\dfrac{\sigma_2 + \sigma_1\rho_{12}}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho_{12}}} \end{pmatrix}.$$

- *(d = 3) Similarly, when* $d = 3$*, let* $\vec{\mu} = (\mu_1, \mu_2, \mu_3), \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ *and*

$$C = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}. \tag{14}$$

Thus, $\left(X_1^*, X_2^*, X_3^*, \ln(\xi_T)\right)$ *follows a* 4*-variate Gaussian distribution with correlation matrix*

$$\widetilde{C} = \left(\begin{array}{ccc|c} 1 & \rho_{12} & \rho_{13} & a_1 \\ \rho_{12} & 1 & \rho_{23} & a_2 \\ \rho_{13} & \rho_{23} & 1 & a_3 \\ \hline a_1 & a_2 & a_3 & 1 \end{array}\right), \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = -\frac{1}{K}\begin{pmatrix} \sigma_1 + \sigma_2\rho_{12} + \sigma_3\rho_{13} \\ \sigma_2 + \sigma_1\rho_{12} + \sigma_3\rho_{23} \\ \sigma_3 + \sigma_1\rho_{13} + \sigma_2\rho_{23} \end{pmatrix}.$$

where $K = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2\rho_{12} + 2\sigma_1\sigma_3\rho_{13} + 2\sigma_2\sigma_3\rho_{23}}$.

4.2.1 Numerical Example

We now provide a numerical example in which we specify the three-dimensional normal distribution that we want to achieve. We assume that its correlation matrix is

$$C = \begin{pmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.2 \\ 0.5 & 0.2 & 1 \end{pmatrix}. \tag{15}$$

For simplicity, we set $T = 1$ and a constant risk-free compounded interest rate $r = 1\%$, and we write the state-price variable ξ_T as in (11) with $\theta = 0.2$ (which would be the case, for example, in a one-dimensional Black-Scholes market with an annual constant drift $\mu = 5\%$ and instantaneous volatility $\sigma = 20\%$, as in this case $\theta = (\mu - r)/\sigma$).

We then apply the numerical procedure described in Proposition 4.2. In Figure 1, Panels E-G, we display scatter plots of X_1, X_2 and X_3 with respect to $\ln(\xi_T)$, which look clearly Gaussian, and we report below each of them $corr\left(X_i, \ln(\xi_T)\right)$. We confirm these findings in the next section, in which we solve explicitly for the multivariate dependence $\left(X_1, X_2, X_3, \ln(\xi_T)\right)$ solving the multivariate cost-efficiency problem (7). In what follows, we verify the results from Proposition 4.6 and the case $(d = 3)$ of Corollary 4.8 in the context of this numerical example. To do so, we report in Table 1 the numerical results obtained by applying our numerical procedure from Proposition 4.2. We observe that the approximate joint distribution obtained as the output of the procedure makes it possible to accurately estimate $corr\left(X_i, \ln(\xi_T)\right)$, $i = 1, 2, 3$. Namely, we observe that these correlation coefficients converge to the theoretical (exact) values as the number of simulations n increases. Furthermore, by using standard statistical tests, we cannot reject that the

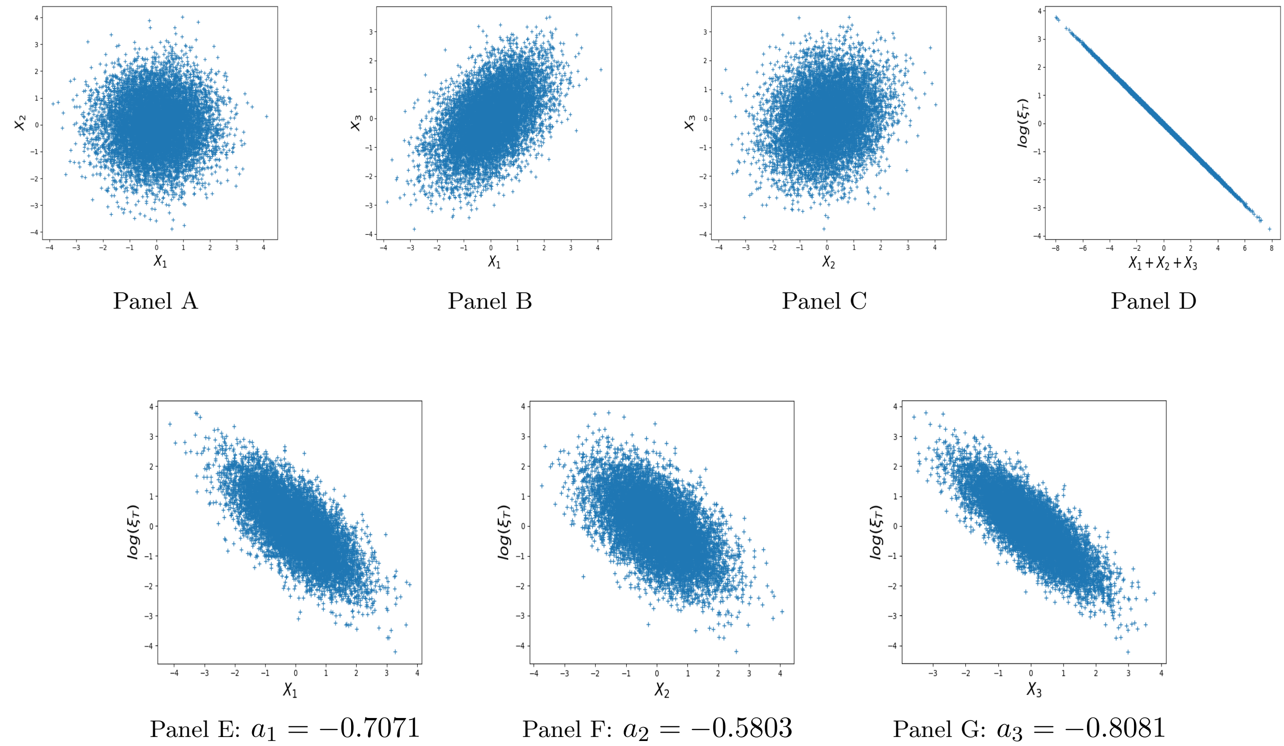

Figure 1: Output dependence of the discrete procedure described in Proposition 4.2 with $n = 10000$ and target dependence at $T = 1$ for (X_1, X_2, X_3) being a trivariate Gaussian distribution with correlation matrix given in (15), vector of means $\vec{\mu} = (0, 0, 0)$, vector of standard deviations $\vec{\sigma} = (1, 1, 1)$ and lognormal (Black-Scholes) market given by ξ_T in (12), with $r = 1\%, \mu = 5\%, \sigma = 20\%$.

dependence is a Gaussian copula.

Number of simulations	$n = 10000$	$n = 100000$	$n = 1000000$	$n = 10000000$	Exact
$corr\big(X_1, \ln(\xi)\big)$	-0.7071	-0.7133	-0.7151	-0.7151	$a_1 = -0.7151$
$corr\big(X_2, \ln(\xi)\big)$	-0.5803	-0.5675	-0.5727	-0.5723	$a_2 = -0.5721$
$corr\big(X_3, \ln(\xi)\big)$	-0.8081	-0.8090	-0.8108	-0.8104	$a_3 = -0.8104$

Table 1: Results from the numerical procedure explained in Proposition 4.2 vs. exact values by Corollary 4.8, case $(d = 3)$. (X_1, X_2, X_3) follow a trivariate Gaussian distribution with correlation matrix given in (15), vector of means $\vec{\mu} = (0, 0, 0)$ and vector of standard deviations $\vec{\sigma} = (1, 1, 1)$. The lognormal (Black-Scholes) market is given by ξ_T in (12), with $T = 1, r = 1\%, \mu = 5\%, \sigma = 20\%$.

As a conclusion from the above experiments, the numerical procedure in Proposition 4.2 provides an accurate approximation for the optimal payoff $(X_1^*, \ldots, X_d^*)$ to (7). The remaining issue is how this payoff is attainable by trading in the assets available in the financial market. We discuss this issue in the following section.

4.2.2 Attainability of Multivariate Cost-Efficient Strategies

The output of the numerical procedure in Proposition 4.2 makes it possible to express the optimal multivariate payoff as a function of the state-price process ξ_T. In a one-dimensional Black-Scholes market, ξ_T is a function of the only stock price S_T. It is then clear that each X_i for $i = 1, \ldots, d$ cannot be a function of S_T and at the same time follow a bivariate Gaussian distribution with $\ln(\xi_T)$, which is also a function of S_T. In this case, the distribution of $\big(X_i, \ln(\xi_T)\big)$ would then be degenerate, as X_i and $\ln(\xi_T)$ are fully (negatively) dependent. However, it is known that fully dependent copulas can approximate any copulas as closely as desired; see Nelsen (2007). Therefore, this does not contradict the fact that the numerical procedure enables us to approximate the optimal solution of (7) at any degree of accuracy.

In the case in which the target distribution is a multivariate Gaussian distribution and the market is lognormal, we have shown in the preceding section that the dependence between ξ_T and $(X_1^*, \ldots, X_d^*)$ is explicit. A natural question is then whether it is possible to construct such a multivariate payoff that has the desired dependence with ξ_T. The answer is affirmative when the market has enough sources of randomness, and we show hereafter that it is possible to build a d-dimensional payoff such that $\big(X_1, \ldots, X_d, \ln(\xi_T)\big)$ follows a multivariate Gaussian distribution with correlation matrix $\widetilde{C}$ given in Proposition 4.6. Note that we easily observe independent variables in a continuous market, e.g., when one of the price processes has independent increments, such as in the Black-Scholes setting.

Proposition 4.9. *Let the target multivariate distribution be a multivariate normal distribution with covariance matrix $\Sigma = (\rho_{ij}\sigma_i\sigma_j T)_{1\leqslant i,j\leqslant d}$ and vector of means $\vec{\mu} = (\mu_1 T, \mu_2 T, \ldots, \mu_d T)$, and let $(W_t)_{t\geqslant 0}$ be a Brownian motion such that $\xi_t = e^{-rt-\frac{\theta^2 t}{2}-\theta W_t}$. The following procedure makes it possible to build a multivariate cost-efficient payoff $\vec{X}^* := (X_1^*, \ldots, X_d^*)$:*

- *Use a Cholesky-type decomposition for the covariance matrix $\Sigma = L \cdot L^\intercal$, where L is a lower triangular matrix such that the column sums are positive, and let $L^\intercal$ denote the transpose of L.*
- *Define $s_j := \sum_{i=1}^{d} L_{ij}$ and $t_k = \sum_{j=1}^{k} \frac{s_j^2 T}{\sum_{j=1}^{d} s_j^2}$, for $k = 1, \ldots, d$. Also, define $\vec{Z}$ as*

$$Z_i := \frac{W_{t_i} - W_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}, \quad i = 1, \ldots, d.$$

- *For $i = 1, \ldots, d$, $\vec{X}^* = L\vec{Z} + \vec{\mu}$.*

By construction, $\vec{X}^*$ has the right joint distribution and the choice of $\vec{Z}$ ensures that $\sum_{i=1}^{d} X_i^*$ is anti-monotonic with $\ln(\xi_T)$*.*

Proof. The target covariance matrix Σ of $\big(X_1 \ldots, X_d\big)$ can be decomposed using Cholesky decomposition as $\Sigma = L^\intercal \cdot L$, where L is the unique lower triangular matrix with positive columns sums, i.e., $s_j := \sum_{i=1}^{d} L_{ij} > 0$ for $j = 1, \ldots, d$. The recursive expressions given by

$$L_{jj} = \pm\sqrt{\Sigma_{jj} - \sum_{k=1}^{j-1} L_{jk}^2}, \qquad L_{ij} = \frac{1}{L_{jj}}\left(\Sigma_{ij} - \sum_{k=1}^{j-1} L_{ik}L_{jk}\right), \quad i > j,$$

enable us to construct this lower triangular matrix.

Recall that for any dates $0 < t_1 < \cdots < t_d = T$, since W_t is a Brownian motion, then

$$Z_i := \frac{W_{t_i} - W_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}, \quad i = 1, \ldots, d,$$

are independent $\mathcal{N}(0,1)$ variables. It is then known that we can use a Cholesky-type decomposition to build the right multivariate Gaussian distribution, namely that $\vec{X} := L\vec{Z} + \vec{\mu}$ follows a multivariate normal distribution with covariance matrix Σ and vector of means $\vec{\mu}$. We thus have full flexibility in choosing the dates $t_1, t_2, \ldots, t_{d-1}$ in the construction of the d independent standard normal variables. We do so hereafter in order to ensure that $\ln(\xi_T)$ is indeed anti-monotonic with the sum of the payoffs X_i, and thus that the result in Theorem 3.1 ensures that our constructed multivariate payoff is multivariate cost-efficient.

Note that we have

$$\sum_{j=1}^{d} X_j = \sum_{j=1}^{d} s_j Z_j + \sum_{j=1}^{d} \mu_j T. \tag{16}$$

Define ω_j as

$$\omega_j := \frac{s_j^2}{\sum_{j=1}^{d} s_j^2} > 0, \tag{17}$$

and observe that $\sum_{j=1}^{d} \omega_j T = T$. For $k = 1, \ldots, d$, define $t_k = \sum_{j=1}^{k} \omega_j T$. By convention, $t_0 = 0$. By definition, we also have that $t_d = T$. Then, observe that

$$\begin{aligned}\sum_{j=1}^{d} s_j Z_j &= \sum_{j=1}^{d} s_j \frac{W_{t_j} - W_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}} = \sum_{j=1}^{d} s_j \frac{W_{t_j} - W_{t_{j-1}}}{\sqrt{\omega_j T}} = \frac{\sqrt{\sum_{j=1}^{d} s_j^2}}{\sqrt{T}} \sum_{j=1}^{d} (W_{t_j} - W_{t_{j-1}}) \\ &= \sqrt{\sum_{j=1}^{d} s_j^2} \frac{W_T}{\sqrt{T}},\end{aligned}$$

where we used (17) and the fact that $s_j > 0$ to simplify. $\sum_{j=1}^{d} X_j$ is thus comonotonic with W_T (and anti-monotonic with $\ln(\xi_T)$). Therefore, this choice of independent variables $\vec{Z}$ ensures that $\vec{X}^*$ is multivariate cost-efficient (Theorem 3.1). In a lognormal market, we indeed have that $\ln(\xi_T) = -\theta W_T - rT - \frac{\theta^2 T}{2}$, which can then be rewritten as

$$\ln(\xi_T) = -\theta\sqrt{T} \frac{\sum_{j=1}^{d} s_j Z_j}{\sqrt{\sum_{j=1}^{d} s_j^2}} - rT - \frac{\theta^2 T}{2}.$$

From (16), we have that

$$\sum_{i=1}^{d} X_i = \sum_{j=1}^{d} s_j Z_j + \sum_{j=1}^{d} \mu_j T = -\frac{\sqrt{\sum_{j=1}^{d} s_j^2}\left(\ln(\xi_T) + rT + \frac{\theta^2}{2} T\right)}{\theta\sqrt{T}} + \sum_{j=1}^{d} \mu_j T. \tag{18}$$

We thus find that there exist $k > 0$ and $k' \in \mathbb{R}$ such that

$$\sum_{i=1}^{d} X_i = -k \ln(\xi_T) + k', \tag{19}$$

where $k = \frac{\sqrt{\sum_{j=1}^{d} s_j^2}}{\theta\sqrt{T}}$ and $k' = -\frac{\sqrt{\sum_{j=1}^{d} s_j^2}\left(r + \frac{\theta^2}{2}\right)\sqrt{T}}{\theta} + \sum_{j=1}^{d} \mu_j T$. Observe that $(X_1, \ldots, X_d)$ has the right multivariate normal distribution (with the right covariance matrix thanks to the Cholesky decomposition) and that its sum is anti-monotonic with $\ln(\xi_T)$. It is thus multivariate cost-efficient (Theorem 3.1). □

Finally, note that by taking the variance of $\sum_{i=1}^d X_i$ in (18) and its original definition as a sum of correlated normal distributions, we have $\sum_{i=1}^d \sigma_i^2 T + 2\sum_{i<j}\rho_{ij}\sigma_i\sigma_j T = \sum_{j=1}^d s_j^2$, and so $\sum_{i=1}^d X_i^* = -k\ln(\xi_T) + k'$ with

$$k = \frac{\sqrt{\sum_{i=1}^d \sigma_i^2 + 2\sum_{i<j}\rho_{ij}\sigma_i\sigma_j}}{\theta}, \quad k' = \sum_{i=1}^d \mu_i T - k\left(rT + \frac{\theta^2 T}{2}\right).$$

From the above derivations, we find that the exact multivariate cost-efficient payoff with multivariate Gaussian distribution does not satisfy (10) and may in fact be path-dependent. Let us provide two examples with $d = 2$ and $d = 3$ for which it is possible to derive the multivariate payoff explicitly.

Example 4.10 ($d = 2$)**.** Let $\vec{\mu} = (\mu_1 T, \mu_2 T), \vec{\sigma} = \left(\sigma_1\sqrt{T}, \sigma_2\sqrt{T}\right)$ and $C = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix}$. Then, in a lognormal market given by (12), the multivariate cost-efficient payoff solving (7) is

$$\begin{cases} X_1^* = \sigma_1\sqrt{T}\frac{W_{t_1}}{\sqrt{t_1}} + \mu_1 T, \\ X_2^* = \sigma_2\rho_{12}\sqrt{T}\frac{W_{t_1}}{\sqrt{t_1}} + \sigma_2\sqrt{1-\rho_{12}^2}\sqrt{T}\frac{W_T - W_{t_1}}{\sqrt{T-t_1}} + \mu_2 T, \end{cases} \tag{20}$$

where $t_1 = \frac{(\sigma_1+\rho_{12}\sigma_2)^2 T}{\sigma_1^2+\sigma_2^2+2\rho_{12}\sigma_1\sigma_2}$.

To prove (20), we apply Proposition 4.9. We thus derive Σ and its Cholesky decomposition and compute the auxiliary values s_i needed to compute t_1:

$$\Sigma = \begin{pmatrix} \sigma_1^2 T & \rho_{12}\sigma_1\sigma_2 T \\ \rho_{12}\sigma_1\sigma_2 T & \sigma_2^2 T \end{pmatrix}, \quad L = \sqrt{T}\begin{pmatrix} \sigma_1 & 0 \\ \rho_{12}\sigma_2 & \sigma_2\sqrt{1-\rho_{12}^2} \end{pmatrix},$$

We find that $s_1 = \sqrt{T}(\sigma_1 + \rho_{12}\sigma_2)$, $s_2 = \sigma_2\sqrt{T}\sqrt{1-\rho_{12}^2}$ and $t_1 = \frac{s_1^2 T}{s_1^2+s_2^2}$ (from (17)).

Example 4.11 ($d = 3$)**.** Let $\vec{\mu} = (\mu_1 T, \mu_2 T, \mu_3 T), \vec{\sigma} = \left(\sigma_1\sqrt{T}, \sigma_2\sqrt{T}, \sigma_3\sqrt{T}\right)$ and C given by (14). Similarly as above, in a lognormal market given by (12), the multivariate cost-efficient payoff solving (7) for this trivariate Gaussian distribution is obtained as

$$\begin{cases} X_1^* = \sigma_1\sqrt{T}\frac{W_{t_1}}{\sqrt{t_1}} + \mu_1 T, \\ X_2^* = \sigma_2\rho_{12}\sqrt{T}\frac{W_{t_1}}{\sqrt{t_1}} + \sigma_2\sqrt{1-\rho_{12}^2}\sqrt{T}\frac{W_{t_2} - W_{t_1}}{\sqrt{t_2-t_1}} + \mu_2 T, \\ X_3^* = \sigma_3\rho_{13}\sqrt{T}\frac{W_{t_1}}{\sqrt{t_1}} + L_{32}\frac{W_{t_2}-W_{t_1}}{\sqrt{t_2-t_1}} + L_{33}\frac{W_T - W_{t_2}}{\sqrt{T-t_2}} + \mu_3 T, \end{cases} \tag{21}$$

where t_1, t_2, L_{32} and L_{33} are obtained using Proposition 4.9:

$$\Sigma = \begin{pmatrix} \sigma_1^2 T & \rho_{12}\sigma_1\sigma_2 T & \rho_{13}\sigma_1\sigma_3 T \\ \rho_{12}\sigma_1\sigma_2 T & \sigma_2^2 T & \rho_{23}\sigma_2\sigma_3 T \\ \rho_{13}\sigma_1\sigma_3 T & \rho_{23}\sigma_2\sigma_3 T & \sigma_3^2 T \end{pmatrix},$$

$$L = \sqrt{T}\begin{pmatrix} \sigma_1 & 0 & 0 \\ \rho_{12}\sigma_2 & \sigma_2\sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13}\sigma_3 & \frac{\rho_{23}\sigma_2\sigma_3 - \rho_{13}\sigma_3\rho_{12}\sigma_2}{\sigma_2\sqrt{1-\rho_{12}^2}} & \frac{\sqrt{\sigma_3^2\sigma_2^2(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})}}{\sigma_2\sqrt{1-\rho_{12}^2}} \end{pmatrix},$$

$$s_1 = \sqrt{T}\left(\sigma_1 + \rho_{12}\sigma_2 + \rho_{13}\sigma_3\right), \quad s_2 = \sqrt{T}\frac{\sqrt{\sigma_2^2\left(1-\rho_{12}^2\right) + \rho_{23}\sigma_2\sigma_3 - \rho_{13}\sigma_3\rho_{12}\sigma_2}}{\sigma_2\sqrt{1-\rho_{12}^2}},$$

$$s_3 = \sqrt{T}\frac{\sqrt{\sigma_3^2\sigma_2^2\left(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}\right)}}{\sigma_2\sqrt{1-\rho_{12}^2}},$$

$$t_0 = 0, \quad t_1 = \frac{s_1^2}{\sum_{j=1}^d s_j^2}T, \quad t_2 = \frac{s_1^2+s_2^2}{\sum_{j=1}^d s_j^2}T, \quad t_3 = T.$$

4.3 Approximation of Multivariate Cost-Efficient Strategies with LogNormal Margins

The previous section describes explicitly the multivariate vector $(X_1^*, \ldots, X_d^*)$ when it has a multivariate normal distribution. These explicit expressions can be used to replicate the multivariate cost-efficient payoff. There are multiple situations in which the normal distribution is a good approximation of other distributions that are more popular in practice. Assume that the target distribution G has lognormal margins. We now look for a vector $(Y_1^*, \ldots, Y_d^*)$ such that $Y_i^*, i = 1, \ldots, d$, is lognormally distributed, and the copula among the Y_i's is the Gaussian copula with homogeneous correlation matrix with parameter ρ. The solution is obtained numerically from the numerical procedure in Algorithm 4.1. We then used the explicit copula in Proposition 4.9 that minimizes the costs with normal margins $(X_i^* := \ln(Y_i^*))$ to construct a candidate multivariate cost-efficient payoff with lognormal margins. Its cost is displayed by the dashed black line in Figure 2; we compare this cost with the numerical approximation of the minimum cost (red line in Figure 2). We make two observations. First, on the one hand, the copula that can be used to construct a multivariate cost-efficient payoff with normal margins does not solve the problem with lognormal margins, as the red and black curves do not match.[9] On the other hand, this copula leads to a very good approximation and can thus offer a practical solution to approximate the multivariate cost-efficient payoff with lognormal margins, which is more in line with the preferences of investors (as it relates to the CRRA utility).

5 Optimal Allocation with Forced Diversification

This final section is dedicated to an application of multivariate cost-efficiency in situations in which an agent (e.g., a manager or a social planner) aims at avoiding a multivariate allocation that would be comonotonic in the various possible dimensions. Here are a few examples.

Consider d agents with respective target probability distribution G_i, $i = 1, \ldots, d$. As pointed out in Section 3, if agents optimize their portfolios separately, their optima are all anti-monotonic with the state-price process, a situation that is often at odds with the objective of a regulator who aims at guaranteeing an overall stable economy in which all agents do not experience worst-case scenarios simultaneously. Similarly, assume that one agent is running a conglomerate. In this case, the agent might be interested in enforcing some diversification among the optimal portfolios of the various business units and therefore a comonotonic allocation is not optimal.

[9] The two curves are closer together when ρ is close to 1. By construction, $\sum_{i=1}^d X_i^*$ is decreasing in ξ_T, though there is no reason why $\sum_{i=1}^d Y_i^* = \sum_{i=1}^d \exp(X_i^*)$ should also be decreasing in ξ_T (condition for cost-efficiency). However, intuitively, when ρ is close to 1 all X_i^*'s behave close to comonotonicity, for which the property would be true.

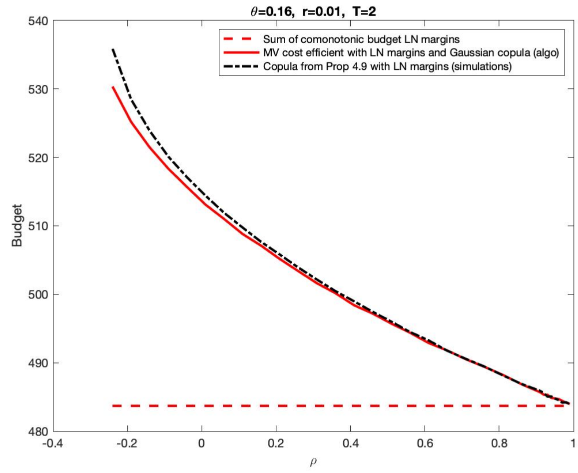

Figure 2: Budget for the MV cost-efficient payoff with lognormal margins and Gaussian copula with homogeneous correlation matrix (with correlation $\rho \geqslant -\frac{1}{d-1}$) as a function of ρ: LN margins with a Gaussian copula. The marginal distributions are lognormal distributions with expected value $\mu_i T = \mathbb{E}[S_T] = 100\exp(\mu T)$ and standard deviation $\sigma_i\sqrt{T} = std(S_T)$. The market is lognormal and given by ξ_T in (12), with $S_0 = 100, T = 2, r = 1\%, \mu = 5\%, \sigma = 25\%$.

In both situations, by using the concept of multivariate cost-efficiency introduced in Section 4, it is possible to enforce a desired dependence at the lowest cost. Such an enforcement is costly and, as a byproduct, we can also quantify the cost of *forced* diversification, which we define as follows.

Consider, for instance, the usual scenario in which some financial institutions in a network are optimizing their investment strategy without adjusting with respect to what other players are doing. Assuming that the d institutions independently optimize a law invariant increasing objective function, then each has an optimal portfolio that is decreasing in the pricing kernel ξ_T. Let us denote these optimal portfolios by X_i^c, with CDF F_i. We have that

$$X_i^c = F_i^{-1}\left(1 - F_{\xi_T}(\xi_T)\right), \quad i = 1, \dots, d.$$

Note that the superscript c stands for **c**omonotonicity, to recall that such allocations are in fact all antimonotonic in $\ln(\xi_T)$ and thus all comonotonic, and that the optimizations are performed independently of each other. To protect the system from collapse due to simultaneous defaults, the regulator could step in and impose an additional constraint in order to control the comonotonicity between the financial institutions. Such an additional constraint would force each institution to depart from its initial strategy. If a regulator imposes some dependence among the institutions that is not the comonotonic copula, but at the same time aims at keeping the same target probability distribution of returns for each institution (and thus the same expected utility), then there is an "extra cost", as the multivariate allocation needs to be achieved in a different way than being antimonotonic with the pricing kernel (thus violating univariate cost-efficiency). Such cost corresponds to that of achieving a multivariate cost-efficient allocation in which the marginal distributions are given by F_i and the copula is not the comonotonic copula, but rather a specified copula (e.g., a Gaussian copula, independence copula...).

Definition 5.1. *The cost of forced diversification is given by*

$$\Delta w_0 := \mathbb{E}\left[\xi_T \sum_{i=1}^{d} X_i^*\right] - w_0,$$

where $(X_1^*, \ldots, X_d^*)$ *is the multivariate cost-efficient allocation and* w_0 *is the total budget spent to achieve these* d *comonotonic allocations, which is simply the sum of the respective budgets to achieve the* d *cost-efficient (univariate) strategies:*

$$w_0 = \sum_{i=1}^{d} \mathbb{E}\left[\xi_T X_i^c\right] = \sum_{i=1}^{d} \mathbb{E}\left[\xi_T F_i^{-1}\left(1 - F_{\xi_T}(\xi_T)\right)\right].$$

Using our approach, one can estimate this additional capital Δw_0 by simply computing the cost of the multivariate cost-efficient allocation obtained using Algorithm 4.1. In the remainder of this section, we compute this extra capital Δw_0 in some special cases of interest. To illustrate this application, we again use the example of the lognormal market in (12). We first look at the most commonly used utility function, and assume that all agents have CRRA utility functions (Section 5.1). In this case, there are no closed-form expressions, and we rely on the discrete procedure described in Section 4.1. We then look at the situation in which all agents have exponential utility functions. In this case, an explicit formula for the extra capital can be derived (Section 5.2). This formula is interesting as it may be used to approximate the case with the CRRA utility. Asymptotic results in which the number of agents becomes arbitrarily large can be investigated using central limit theorems.

Once the extra amount of capital needed is computed, a few additional questions arise: who should bear this additional cost? Is there a way to "implement" this strategy in practice? The advantage of having d strategies that are not all comonotonic with the pricing kernel (which, we recall, is unique in our setting) is that all agents will not experience losses at the same time, thus potentially reducing the overall amount of risk. Nonetheless, it is not clear a priori how the regulator should incentivize banks to adapt their investment strategies as a function of other banks' behavior. In this direction, our suggested practical approach would work as follows. An external entity (say, a guarantee fund) collects the amount of money Δw_0 to pay for this protection, for instance by imposing an additional capital requirement on financial institutions (namely $\mathbb{E}[\xi_T X_i^*] - \mathbb{E}[\xi_T X_i^c]$ for the ith institution). This amount would then need to be invested so as to reach the desired purpose, i.e. to obtain

$$\sum_{i=1}^{d} X_i^* - \sum_{i=1}^{d} X_i^c.$$

5.1 Case when All Agents Have CRRA Utility Function - LogNormal Margins

Consider d agents maximizing their respective CRRA utility with risk aversion parameter γ_i and budget w_i. The solution to each individual maximization problem

$$\max_{X_i} \mathbb{E}\left[U_{\gamma_i}(X_i)\right], \quad \text{s.t. } \mathbb{E}\left[\xi_T X_i\right] = w_i,$$

where

$$U_\gamma(x) = \begin{cases} \dfrac{x^{1-\gamma}}{1-\gamma}, & \gamma > 0, \gamma \neq 1, \\ \ln(x), & \gamma = 1, \end{cases}$$

is given by

$$X_i^c = \xi_T^{-\frac{1}{\gamma_i}} \exp\left(\left(1 - \frac{1}{\gamma_i}\right)\left(rT + \frac{\theta^2 T}{2\gamma_i}\right)\right) w_i. \tag{22}$$

It is easy to show that the quantity in (22) follows a lognormal distribution

$$F_i := \log\mathcal{N}\left(rT + \log(w_i) + \frac{\theta^2 T}{\gamma_i} - \frac{\theta^2 T}{2\gamma_i^2}, \frac{\theta\sqrt{T}}{\gamma_i}\right), \quad i = 1, \dots, d. \tag{23}$$

Now suppose that these agents are interested in achieving a payoff that has expected value $\mu_i T$ and standard deviation $\sigma_i\sqrt{T}$. To do so, the budget that they would need to invest is[10]

$$w_i = e^{-rT}\mu_i T \exp\left(-\theta\sqrt{T}\sqrt{\log\left(1 + \frac{\sigma_i^2 T}{(\mu_i T)^2}\right)}\right). \tag{24}$$

Using this expression of the budget needed to achieve F_i, the following proposition provides the extra budget $\Delta w_0 := \mathbb{E}\left[\xi_T \sum_{i=1}^d X_i^*\right] - \sum_{i=1}^d \mathbb{E}\left[\xi_T F_i^{-1}(1 - F_{\xi_T}(\xi_T))\right]$ needed for diversification.

Proposition 5.2. *Let $w^* := \mathbb{E}\left[\xi_T \sum_{i=1}^d X_i^*\right]$ be the minimum cost to obtain a d-tuple of lognormal distributions (optimal for CRRA investors) with means $\vec{\mu} = (\mu_1 T, \dots, \mu_d T)$, standard deviations $\vec{\sigma} = (\sigma_1\sqrt{T}, \dots, \sigma_d\sqrt{T})$ and Gaussian copula with correlation matrix $(\rho_{ij})_{1\leqslant i,j\leqslant d}$. This cost can be computed via the numerical procedure described in Section 4.1. Thus, the extra budget Δw_0 needed in this case is*

$$\Delta w_0 \quad = \quad w^* - e^{-rT}\sum_{i=1}^d \mu_i T \exp\left(-\theta\sqrt{T}\sqrt{\log\left(1 + \frac{\sigma_i^2 T}{(\mu_i T)^2}\right)}\right), \tag{25}$$

which is equal to 0 when all $\rho_{ij} = 1$, $i, j = 1, \dots, d$ (comonotonicity).

An illustration of this extra budget Δw_0 computed in Proposition 5.2 is displayed in Figure 3 in the case of a Gaussian copula that has an homogeneous matrix of correlation coefficients (all ρ_{ij} are identically equal to the parameter ρ). In particular, note that the extra budget tends to 0 as the correlation ρ tends to 1 and the cost of achieving independence ($\rho = 0$) increases with the number of dimensions.

As there is no closed-form expression for the budget to achieve a cost-efficient multivariate distribution with lognormal margins and Gaussian copula, we estimate this budget numerically.[11] In particular, we note from Figure 3 that, as expected, the extra budget tends to 0 as the correlation ρ tends to 1. Note also that the cost of achieving independence ($\rho = 0$) increases with the number of dimensions. The case when

[10] The expression (24) follows from the expressions of the mean and the variance, i.e., $\mu_i T = w_i \exp\left(rT + \frac{\theta^2 T}{\gamma_i}\right)$, which implies that $\gamma_i = \theta^2 T\left(\log(\mu_i T) - rT - \log(w_i)\right)^{-1}$. Furthermore, $\sigma_i^2 T = w_i^2 \exp\left(2rT + \frac{2\theta^2 T}{\gamma_i} + \frac{\theta^2 T}{\gamma_i^2} - 1\right)$. Thus, replacing γ_i into this second equation and rearranging leads to (24).

[11] We proceed as follows. First, we simulate $N = 10'000'000$ samples of the multivariate distribution $(X_1, \dots, X_d)$. We then compute $S = \sum_{i=1}^d X_i$ and its empirical CDF F_S. We then obtain ξ_T as $F_{\xi_T}^{-1}(1 - F_S(S))$. Then the budget w^* is simply the Monte Carlo estimate of $\mathbb{E}[\xi_T S]$.

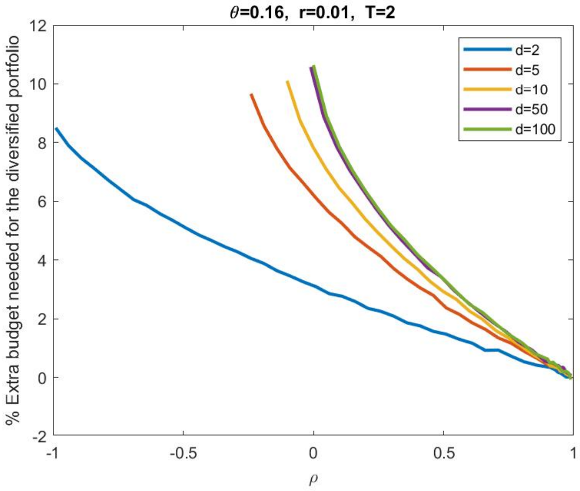

Figure 3: Additional budget (in % of the cost of the comonotonic allocation) due to forced diversification: LN margins with a Gaussian copula with homogeneous correlation matrix with parameter ρ (with $\rho \geqslant -\frac{1}{d-1}$). The marginal distributions are lognormal distributions with expected value $\mu_i T = \mathbb{E}[S_T] = 100\exp(\mu T)$ and $\sigma_i\sqrt{T} = std(S_T)$. The market is lognormal and given by ξ_T in (12), with $S_0 = 100, T = 2, r = 1\%, \mu = 5\%, \sigma = 25\%$.

all agents have CRRA utility functions, or equivalently, when the target joint distribution has lognormal margins, cannot be solved explicitly. In what follows, we study the case of normal margins, corresponding to exponential utility maximizers. In that case, Δw_0 can be interpreted directly as a closed-form expression is available (see Proposition 5.3 hereafter and Remark 5.4 for an interpretation).

5.2 Case when All Agents Have Exponential Utility Function - Normal Margins

Consider d agents maximizing their respective exponential expected utility with risk aversion parameter $\alpha_i > 0$ and budget w_i. The solution to each individual maximization problem

$$\max_{X_i} \mathbb{E}\left[-e^{-\alpha_i X_i}\right], \quad \text{s.t. } \mathbb{E}\left[\xi_T X_i\right] = w_i, \tag{26}$$

is given by

$$X_i^c = w_i\, e^{rT} - \frac{1}{\alpha_i}\left(\ln(\xi_T) + rT - \frac{\theta^2 T}{2}\right), \quad i = 1, \dots, d. \tag{27}$$

Respectively, these d exponential utility maximizers have an optimal allocation that is anti-monotonic to ξ_T (and so the d optimal allocations are comonotonic), which also solves the one dimensional cost-efficiency problem by Lemma 2.1, i.e., $X_i^* := F_i^{-1}\big(1 - F_{\xi_T}(\xi_T)\big)$, where F_i denotes the distribution of X_i^c for $i = 1, \dots, d$ and is a normal distribution:

$$F_i := \mathcal{N}\left(w_i e^{rT} + \frac{\theta^2 T}{\alpha_i}, \frac{\theta\sqrt{T}}{\alpha_i}\right), \quad i = 1, \dots, d;$$

for details, see Bernard et al. (2015). Using the notation from the previous section, this corresponds to an annualized expected return and volatility given by

$$\mu_i T = w_i e^{rT} + \frac{\theta^2 T}{\alpha_i}, \quad \sigma_i \sqrt{T} = \frac{\theta\sqrt{T}}{\alpha_i}, \tag{28}$$

so that the budget needed to achieve a normal distribution $\mathcal{N}\left(\mu_i T, \sigma_i\sqrt{T}\right)$ is

$$w_i = e^{-rT}(\mu_i - \sigma_i\theta)T. \tag{29}$$

In the proposition below, we compute the cost of achieving an optimal payoff that follows a multivariate Gaussian distribution with given parameters. We then discuss the cost of diversification for this case, that is, what additional budget an agent would need to obtain such a "diversified" portfolio (constrained optimum as in Proposition 4.9) with respect to an unconstrained optimum (comonotonic allocation as in Theorem 2.2)

Proposition 5.3. *The minimum cost of obtaining a multivariate normal distribution with covariance matrix* Σ*, vector of means* $\vec{\mu} = (\mu_1 T, \dots, \mu_d T)$ *and standard deviations* $\vec{\sigma} = (\sigma_1\sqrt{T}, \dots, \sigma_d\sqrt{T})$ *(cf. (28)) is given by*

$$\mathbb{E}\left[\xi_T \sum_{i=1}^d X_i^*\right] = e^{-rT}\left(\sum_{i=1}^d \mu_i T - \theta T\sqrt{\sum_{i=1}^d \sigma_i^2 + 2\sum_{i<j}\rho_{ij}\sigma_i\sigma_j}\right).$$

Thus, the extra budget needed in this case is

$$\Delta w_0 \;=\; e^{-rT}\left(\sum_{i=1}^d \sigma_i - \sqrt{\sum_{i=1}^d \sigma_i^2 + 2\sum_{i<j}\rho_{ij}\sigma_i\sigma_j}\right)\theta T, \tag{30}$$

which is equal to 0 in the comonotonic situation in which all $\rho_{ij} = 1$, $i, j = 1, \dots, d$.

Proof. By Theorem 3.1, we know that the multivariate cost-efficient payoff $\vec{X}^*$ is such that the sum of its elements is anti-monotonic with $\ln(\xi_T)$: $\sum_{i=1}^d X_i^* = f(\ln(\xi_T))$, for some non-increasing measurable function f. More precisely, it can be shown that in the Gaussian case f is affine:[12]

$$\sum_{i=1}^d X_i^* = -k\ln(\xi_T) + k',$$

where k and k' are given as follows

$$k = \frac{\sqrt{\sum_{i=1}^d \sigma_i^2 + 2\sum_{i<j}\rho_{ij}\sigma_i\sigma_j}}{\theta}, \; k' = \sum_{i=1}^d \mu_i T - \frac{\sqrt{\sum_{i=1}^d \sigma_i^2 + 2\sum_{i<j}\rho_{ij}\sigma_i\sigma_j}}{\theta}\left(rT + \frac{\theta^2 T}{2}\right). \tag{31}$$

[12] The argument works as follows: let $X \sim \mathcal{N}(a, b), Y \sim \mathcal{N}(c, d)$, with $a, c \in \mathbb{R}, b, d > 0$, and consider $Y = f(X)$, for some non-increasing measurable function f. We have

$$F_Y(x) = \mathbb{P}(Y \leqslant x) = \mathbb{P}(f(X) \leqslant x) = \mathbb{P}(X \geqslant f^{-1}(x)) = \Phi\left(\frac{-f^{-1}(x) + a}{b}\right).$$

However, by definition, it also holds that $F_Y(x) = \Phi\left(\frac{x-c}{d}\right)$, which implies that $f(x) = -\frac{d}{b}(x-a) + c$.

Using the fact that $\mathbb{E}\left[\xi_T \ln(\xi_T)\right] = e^{-rT}\left(-rT + \frac{\theta^2 T}{2}\right)$, the cost of $\sum_{i=1}^d X_i^*$ is computed as follows:

$$\mathbb{E}\left[\xi_T \sum_{i=1}^d X_i^*\right] = e^{-rT} k \left(rT - \frac{\theta^2 T}{2}\right) + e^{-rT} k' = e^{-rT}\left(\sum_{i=1}^d \mu_i T - \theta T \sqrt{\sum_{i=1}^d \sigma_i^2 + 2\sum_{i<j} \rho_{ij}\sigma_i\sigma_j}\right).$$

Considering (29), the extra budget $\Delta w_0 := \mathbb{E}\left[\xi_T \sum_{i=1}^d X_i^*\right] - \mathbb{E}\left[\xi_T \sum_{i=1}^d F_i^{-1}(1 - F_{\xi_T}(\xi_T))\right]$ needed to "force" some diversification and to deviate from comonotonicity can then be easily computed as

$$\Delta w_0 = e^{-rT}\left(\sum_{i=1}^d \mu_i T - \theta T \sqrt{\sum_{i=1}^d \sigma_i^2 + 2\sum_{i<j} \rho_{ij}\sigma_i\sigma_j} - \sum_{i=1}^d (\mu_i - \sigma_i\theta) T\right).$$

We then obtain (30), which ends the proof. □

Remark 5.4. Note that the expression for the extra cost in (30) is proportional to the difference between the standard deviation of the comonotonic sum (maximum possible standard deviation for a sum of variables with given marginal distributions), which is simply the sum of the standard deviations $\sqrt{T}\sum_{i=1}^d \sigma_i$, and the standard deviation of a correlated sum of variables $\sqrt{T}\sqrt{\sum_{i=1}^d \sigma_i^2 + 2\sum_{i<j}\rho_{ij}\sigma_i\sigma_j}$.

5.3 Systemic Risk Example

In this section, we develop a concrete example of how systemic risk can be significantly reduced. This is an oversimplified illustration of the financial system and results are reported here to provide the magnitude of the impact of our proposed enforced diversification. Obtaining accurate estimates of these measures based on empirical inputs is beyond the scope of our paper.

Assume a complete lognormal market with state price process ξ_T defined in (11), and consider d financial institutions that are maximizing a CRRA utility function with respective risk aversion coefficient γ_i. Each institution has then optimal assets at time T as in (22):

$$X_i = w_i\, \xi_T^{-\frac{1}{\gamma_i}} \exp\left(\left(1 - \frac{1}{\gamma_i}\right)\left(rT + \frac{\theta^2 T}{2\gamma_i}\right)\right), \tag{32}$$

in which w_i denotes the initial assets' value. Furthermore, they are all antimonotonic with the state-price process (as a result of the institutions optimizing their own univariate expected utility function).

Each institution has a probability of default that can be estimated, for instance, from the credit ratings. For simplicity, we assume a default probability of 0.09% for each of the d institutions (corresponding to the average annual default rate among A-rated issuers over the period 1920-2018). Let $\mathcal{D}_i$ denote the default event. We can then find the liabilities threshold d_i in the Merton model such that the probability of default is 0.09%:

$$\mathcal{D}_i := \{X_i < d_i\}, \quad \text{with } d_i = F_i^{-1}(0.09\%),$$

where F_i is the CDF given in (23) of the lognormally distributed assets X_i.

The regulator of the financial system is concerned with the total loss left for the society,

$$L := \sum_{i=1}^{d} (d_i - X_i)^+ , \tag{33}$$

resulting from the possible bankruptcy of the d financial institutions. In addition, the regulator may also want to control the joint probability of default of these institutions,

$$\mathcal{JD} := \mathbb{P}\left(\bigcap_{i=1}^{d} \mathcal{D}_i\right), \tag{34}$$

and to find mechanisms to reduce it.

Without any government intervention, the optimal portfolios (32) are all decreasing in ξ_T and thus comonotonic. As a result, the pairwise correlation between the assets of any two institutions is maximized, and the joint probability of default of d institutions that each have a probability 0.09% to default is also equal to

$$\mathcal{JD} = 0.09\%,$$

as if one defaults they all default simultaneously because of comonotonicity.

Suppose that the regulator aims to avoid this situation, and wants that the d financial institutions hold assets $(X_1^*, \ldots, X_d^*)$ such that each X_i^* has the same lognormal distribution as X_i in (32) (so that the individual probability of default is unchanged) but in such a way that the copula of $(X_1^*, \ldots, X_d^*)$ is a Gaussian copula with correlation parameter $\rho < 1$ (the case $\rho = 1$ correspond to the comonotonicity situation discussed above). We then assume that there is a government intervention forcing each financial institution to hold an additional position $X_i^* - X_i$, so that the assets of company i are given by

$$X_i + (X_i^* - X_i) = X_i^*.$$

Such move from holding X_i^* instead of X_i is costly, specifically the extra cost needed is exactly the cost of diversification computed in the previous section:

$$\Delta w_0(\rho) := \sum_{i=1}^{d} \mathbb{E}\left[\xi_T (X_i^* - X_i)\right],$$

which is given in Proposition 5.2 in (25) and which can be estimated numerically via our algorithm. We report the values in the first row of Table 2. In the second row, we report the joint probability of default.

ρ	$-\frac{1}{d-1} = -0.25$	-0.125	0	0.25	0.5	0.75	1
Δw_0	7.7	5.9	4.8	3.2	2	0.94	0
$\mathcal{JD}$	0	0	3.3e-06	0.00037	0.0031	0.011	0.029
$\mathbb{E}(L^*)$	7.7	6	4.9	3.3	2	0.99	0.045
$std(L^*)$	0.25	0.25	0.25	0.27	0.31	0.4	0.57

Table 2: $d = 5$ financial institutions maximizing a CRRA utility function with risk aversion coefficient γ_i (respectively equal to 0.3, 0.4, 0.5, 0.6, 0.7 and 0.8). The market is lognormal and given by ξ_T in (12), with $T = 2, r = 1\%, \mu = 5\%, \sigma = 25\%$. In addition, we assume that $w_i = 20$ so that $\sum_i w_i = 100$.

There is clearly a trade-off between the extra cost needed to intervene and the joint probability of default, and of the distribution of the loss for society L, which after intervention becomes

$$L^*(\rho) := \sum_{i=1}^{d} (d_i - X_i^*)^+ + \Delta w_0(\rho), \tag{35}$$

in which $L^*(1) = L$ is defined in (33). To help the regulator design her intervention, we assume that she aims at maximizing her expected utility

$$g(\rho) := \mathbb{E}\left[v\left(-L^*(\rho)\right)\right], \tag{36}$$

where $v(x) = \frac{x^{1-\eta}}{1-\eta}$ is a CRRA utility function with parameter $\eta \in [0, 1)$. On the one hand, if the regulator is close to the risk neutral situation $\eta = 0$, then she prefers to not intervene, and her utility will be maximized at $\rho = 1$ for which $\Delta w_0(\rho = 1) = 0$ and the loss for society L is the original loss given in (33). On the other hand, when she is risk averse $\eta \in (0, 1)$, then there is an optimal level of correlation ρ for which her expected utility is maximized as can be seen from Figure 4. The more risk averse she is, the more protection she is willing to purchase, as it can be seen from the extra budget cost computed in the first row of Table 2.

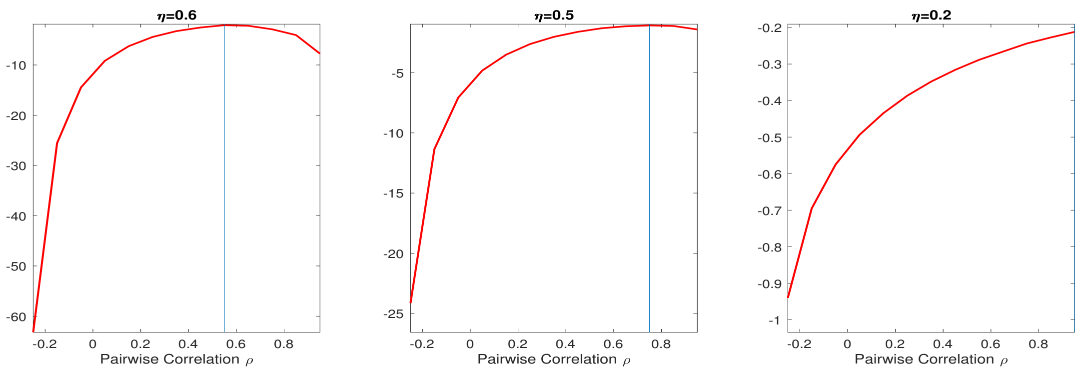

Figure 4: Expected utility (36) of the regulator as a function of ρ for various choices of the risk aversion parameter: $\eta = 0.2$, $\eta = 0.5$ and $\eta = 0.6$. Same assumptions as in Table 2: $d = 5$ financial institutions maximizing a CRRA utility function with risk aversion coefficient γ_i (respectively equal to 0.1, 0.3, 0.5, 0.7 and 0.9). The market is lognormal and given by ξ_T in (12), with $T = 2, r = 1\%, \mu = 5\%, \sigma = 25\%$. We also set $w_i = 20$ so that $\sum_i w_i = 100$.

6 Conclusions

This paper generalizes the concept of univariate cost-efficiency (Dybvig (1988a,b), Bernard et al. (2014)) to a multivariate setting and shows how this can be used to give a necessary condition for a multivariate allocation to optimize a multivariate objective that is law invariant and increasing in at least one of the components. We find that the optimal multivariate portfolio for supermodular preferences must exhibit comonotonicity. It appears more challenging, on the other hand, to establish the optimal allocation in case of submodular preferences. However, we show that this optimum must be multivariate cost-efficient.

Furthermore, for every given joint distribution, we are able to derive the payoff that yields this distribution at cheapest possible cost (multivariate cost-efficient payoff). We believe that this is a sensible approach to optimal multivariate portfolio choice and illustrate this with an application to the management of systemic risk. We expect that the characterization of the optimal multivariate cost-efficient payoffs is useful in designing an efficient algorithm to search for solutions to multivariate optimization problems of the form (8), and thus in extending the results obtained in the univariate setting in Bernard et al. (2019a). In particular, the analysis presented in this paper allows to deal with "law-invariant settings" and the absence of additional risk constraints on the multivariate investment. For example, an interesting research direction is to develop a setting that allows to solve optimal investments of pension funds that not only have a desired targeted multivariate distribution but also specific scheduled liability commitments in the future corresponding to future pension payments. Such additional constraints are so-called "state-dependent".

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Appendix

A Proof of Theorem 2.2

Assume that an optimum exists and denote it by $(X_1^*,\dots,X_d^*)$. It can be shown that this solution must have budget w_0, that is, $\mathbb{E}\left[\xi_T\sum_{i=1}^d X_i^*\right]=w_0$, otherwise the optimality of $(X_1^*,\dots,X_d^*)$ can be violated. Indeed, assume that the budget w_0' to achieve $(X_1^*,\dots,X_d^*)$ satisfies $w_0'<w_0$, and consider the allocation $(X_1^*,\dots,X_{j_0}^*+(w_0-w_0')e^{rT},\dots,X_d^*)$. This allocation requires exactly a budget w_0 but has a strictly higher expected utility, as the utility U is strictly increasing in x_{j_0}, thus violating the optimality of $(X_1^*,\dots,X_d^*)$.

Furthermore, note that the maximum expectation of a supermodular function over all random vectors $(X_1,\dots,X_d)$ with given margins is achieved with the comonotonic allocation $(X_1^c,\dots,X_d^c)$ (Lorentz (1953); see also Puccetti and Rüschendorf (2015)). Let F_i^* denote the CDF of X_i^* and define

$$(X_1^c,\dots,X_d^c):=\left((F_1^*)^{-1}(\mathcal{U}),\dots,(F_d^*)^{-1}(\mathcal{U})\right),$$

for some uniform variable $\mathcal{U}$ over $(0,1)$. We then have

$$\mathbb{E}\left[U(X_1^*,\dots,X_d^*)\right]\leqslant\mathbb{E}\left[U(X_1^c,\dots,X_d^c)\right]. \tag{37}$$

This inequality holds in particular for the choice of uniform $\mathcal{U}=1-F_{\xi_T}(\xi_T)$ (which is uniform as ξ_T is continuously distributed). Thus, from Lemma 2.1,

$$\mathbb{E}\left[\xi_T X_i^c\right]\leqslant\mathbb{E}\left[\xi_T X_i^*\right],\quad i=1,\dots,d,$$

for X_i^c and X_i^* have the same distribution, but X_i^c is anti-monotonic with ξ_T by construction. Therefore,

$$\mathbb{E}\left[\xi_T\sum_{i=1}^d X_i^c\right]\leqslant\mathbb{E}\left[\xi_T\sum_{i=1}^d X_i^*\right]=w_0. \tag{38}$$

We thus have that $(X_1^c,\dots,X_d^c)$ provides a higher expected utility (37) for a lower budget (38), which violates the optimality of $(X_1^*,\dots,X_d^*)$ of the expected utility maximization (5). Then we must have $\mathbb{E}\left[U(X_1^c,\dots,X_d^c))\right]=\mathbb{E}\left[U(X_1^*,\dots,X_d^*)\right]$. From Lemma 2.1, by the a.s. uniqueness of the minimum cost strategy that achieves a given CDF, (38) is a strict inequality unless, for each i,

$$X_i^*=(F_i^*)^{-1}\left(1-F_{\xi_T}(\xi_T)\right)\ a.s. \tag{39}$$

However, if the inequality is strict, this would imply that the budget constraint is not binding at the optimum, which again contradicts the optimality of $(X_1^*,\dots,X_d^*)$. Therefore, the optimal solution to the maximum expected utility problem $(X_1^*,\dots,X_d^*)$ must almost surely be a comonotonic vector, and (39) must hold for all $i=1,\dots,d$. $\square$

B CRRA Bivariate Utility

In this section we provide an example of a two-dimensional utility maximization problem the solution of which can be obtained in quasi-closed form.

Let $d=2$ and consider the following additive utility function supported on $\mathbb{R}_+^2$:

$$U_\gamma(X_1,X_2) = U_{\gamma_1}(X_1) + U_{\gamma_2}(X_2),$$

where $U_{\gamma_1}, U_{\gamma_2}$ are CRRA power utility functions on $\mathbb{R}_+$ with respective risk aversion parameter $\gamma_i, i=1,2$; cf. definition in Section 5.1.

For tractability, we assume again a lognormal market as in Section 4.2.

Proposition B.1. *The a.s. unique optimal solution of the problem*

$$\max_{(X_1,X_2)\in A^{w_0}} \mathbb{E}\left[U_{\gamma_1}(X_1) + U_{\gamma_2}(X_2)\right],$$

where $A^{w_0} := \{(X_1,X_2) : \mathbb{E}\left[\xi_T(X_1+X_2)\right] = w_0\}$, w_0 *being the initial budget, is given by*

$$\begin{pmatrix} X_1^* \\ X_2^* \end{pmatrix} = \begin{pmatrix} (\xi_T\lambda_1^*)^{-\frac{1}{\gamma_1}} \\ (\xi_T\lambda_2^*)^{-\frac{1}{\gamma_2}} \end{pmatrix},$$

where

$$\lambda_1^* = \left(\frac{\exp\left(\left(rT+\frac{\theta^2T}{2\gamma_1}\right)\left(\frac{1}{\gamma_1}-1\right)\right)}{\lambda^* w_0}\right)^{\gamma_1}, \quad \lambda_2^* = \left(\frac{\exp\left(\left(rT+\frac{\theta^2T}{2\gamma_2}\right)\left(\frac{1}{\gamma_2}-1\right)\right)}{(1-\lambda^*) w_0}\right)^{\gamma_2},$$

and λ^* *is the unique solution of the following equation:*

$$\frac{\exp\left(\left(rT+\frac{\theta^2T}{2\gamma_1}\right)\left(\frac{1}{\gamma_1}-1\right)\right)w_0^{1-\gamma_1}}{\exp\left(\left(rT+\frac{\theta^2T}{2\gamma_2}\right)\left(\frac{1}{\gamma_2}-1\right)\right)w_0^{1-\gamma_2}} = \frac{\lambda^{\gamma_1}}{(1-\lambda)^{\gamma_2}}. \tag{40}$$

Proof. Let $\lambda_1,\lambda_2>0$ be Lagrange multipliers. For each $\omega\in\Omega$, consider the following auxiliary problem:

$$\max_{(x_1,x_2)} U_{\gamma_1}(x_1) + U_{\gamma_2}(x_2) - \lambda_1\xi_T(\omega)x_1 - \lambda_2\xi_T(\omega)x_2.$$

Imposing a first-order optimality condition, it follows that

$$\begin{pmatrix} X_1^* \\ X_2^* \end{pmatrix} = \begin{pmatrix} (\lambda_1\,\xi_T(\omega))^{-\frac{1}{\gamma_1}} \\ (\lambda_2\,\xi_T(\omega))^{-\frac{1}{\gamma_2}} \end{pmatrix},$$

where $\lambda_1,\lambda_2>0$ are such that $\mathbb{E}\left[\xi_T X_1^*\right] = X_{0,1}$ and $\mathbb{E}\left[\xi_T X_2^*\right] = X_{0,2}$, respectively, with $X_{0,1}+X_{0,2}=w_0$.

Plugging (42) in the budget equations gives

$$\lambda_1^* = \frac{\exp\left(\left(rT + \frac{\theta^2 T}{2\gamma_1}\right)\left(1 - \frac{1}{\gamma_1}\right)\right)}{X_{0,1}^{\gamma_1}}, \quad \lambda_2^* = \frac{\exp\left(\left(rT + \frac{\theta^2 T}{2\gamma_2}\right)\left(1 - \frac{1}{\gamma_2}\right)\right)}{X_{0,2}^{\gamma_2}}. \tag{41}$$

The last step is to find the optimal allocation $(X_{0,1}, X_{0,2})$. To do so, we introduce a parameter $\lambda \in (0,1)$ such that $X_{0,1} = \lambda w_0$ and $X_{0,2} = (1-\lambda)w_0$. Then, we maximize $\mathbb{E}\left[U_{\gamma_1}(X_1^*)\right] + \mathbb{E}\left[U_{\gamma_2}(X_2^*)\right]$ with respect to the optimal allocation coefficient λ. After standard calculations, we obtain that λ^* uniquely satisfies the implicit equation (40), which can be easily solved numerically. There is thus a unique budget split $X_{0,1}^* = \lambda^* w_0$ and $X_{0,2}^* = (1-\lambda^*)w_0$ so that using the expression (41), we find that

$$\begin{pmatrix} X_1^* \\ X_2^* \end{pmatrix} = \begin{pmatrix} X_{0,1}^* \, \xi_T(\omega)^{-\frac{1}{\gamma_1}} \exp\left(\left(rT + \frac{\theta^2 T}{2\gamma_1}\right)\left(1 - \frac{1}{\gamma_1}\right)\right) \\ X_{0,2}^* \, \xi_T(\omega)^{-\frac{1}{\gamma_2}} \exp\left(\left(rT + \frac{\theta^2 T}{2\gamma_2}\right)\left(1 - \frac{1}{\gamma_2}\right)\right) \end{pmatrix}, \tag{42}$$

This ends the proof. □

Remark B.2. We draw here a connection between the result in Proposition B.1 and the solution of a univariate CRRA utility maximization problem

$$\max_{Y} \mathbb{E}\left[U_\gamma(Y)\right], \quad \text{s.t. } \mathbb{E}\left[\xi_T Y\right] = w_0,$$

which is explicitly given by

$$Y^* = w_0 \, \xi_T^{-\frac{1}{\gamma}} \exp\left(\left(rT + \frac{\theta^2 T}{2\gamma}\right)\left(1 - \frac{1}{\gamma}\right)\right). \tag{43}$$

It is then easy to observe that the expression in (42), combined with (41), corresponds to the solution of two separate univariate CRRA utility maximization problems with fixed, individual budgets $X_{0,1}^*$ and $X_{0,2}^*$ and respective risk aversion coefficients γ_1 and γ_2.

C Proof of Theorem 3.1

Denote by $\left(X_1, \ldots, X_d\right)$ an optimal solution to (7), which is assumed to exist. Problem (7) can also be written as

$$\min_{S \sim H} \mathbb{E}\left[\xi_T S\right],$$

which is the standard cost-efficiency problem in one dimension in which H is a one-dimensional distribution. The proof now follows that from Bernard et al. (2014), and $\mathbb{E}\left[\xi_T S\right]$ is minimal whenever (S, ξ_T) is an anti-monotonic pair. □

D Proof of Proposition 4.6

We want to solve the multivariate cost-efficiency problem (7) where G is the multivariate Gaussian distribution. The correlation matrix of $(X_1, \dots, X_d, \xi_T)$ is given by

$$\widetilde{\mathbf{C}} = \left(\begin{array}{cccc|c} 1 & \rho_{12} & \cdots & \rho_{1d} & a_1 \\ \rho_{12} & \ddots & \ddots & \rho_{2d} & a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \rho_{d1} & \rho_{d2} & \cdots & 1 & a_d \\ \hline a_1 & a_2 & \cdots & a_d & 1 \end{array}\right). \tag{44}$$

Define the vector $\vec{a} = (a_1, \dots, a_d)$ as the vector of correlations between $\ln(\xi_T)$ and X_i, $i = 1, \dots, d$. Let $\xi_T \sim \log\mathcal{N}(\mu_{\xi_T}, \sigma_{\xi_T})$, $X_i \sim \mathcal{N}(\mu_i, \sigma_i)$ and $(\ln(\xi_T), X_i)$, for $i = 1, \dots, d$, follow a bivariate Gaussian distribution with vector of means $\vec{\mu} = (\mu_{\xi_T}, \mu_i)$, vector of standard deviations $\vec{\sigma} = (\sigma_{\xi_T}, \sigma_i)$ and correlation ρ_{ξ_T, X_i}. After some calculations, we obtain

$$\mathbb{E}[\xi_T X_i] = (\mu_i + \sigma_{\xi_T}\rho_{\xi_T, X_i}\sigma_i)\, e^{-rT}.$$

Now, let us define $b_i := \theta\sqrt{T}\, e^{-rT}\sigma_i$ and $\widetilde{\mu}_i := e^{-rT}\mu_i, i = 1, \dots, d$. The multivariate cost-efficiency problem (7) can then be expressed as

$$\min_{a_1, \dots, a_d} \sum_{i=1}^{d} (\widetilde{\mu}_i + b_i a_i), \tag{45}$$

subject to $\widetilde{C}$ being a valid correlation matrix. Since $\widetilde{\mu}_i, i = 1, \dots, d$, are fixed, solving (45) is in turn equivalent to solving $\min_{a_1, \dots, a_d} \sum_{i=1}^{d} b_i a_i$.

Let us denote by $C = (\rho_{ij})_{1 \leqslant i,j \leqslant d}$ the correlation matrix among the X_i's. By taking the Cholesky decomposition $C = LL^\intercal$, we can rewrite $\widetilde{C} = \begin{pmatrix} LL^\intercal & L\vec{k} \\ \vec{k}^\intercal L^\intercal & \vec{k}^\intercal\vec{k} \end{pmatrix}$, for some vector $\vec{k}$ such that $\vec{a} = \vec{k}^\intercal L^\intercal$. This leads to the following simplified version of the problem in (45):

$$\min_{k_1, \dots, k_d} \sum_{i=1}^{d} c_i k_i, \qquad \text{s.t.} \sum_{i=1}^{d} k_i^2 = 1, \tag{46}$$

where

$$\vec{c} = (c_1, \dots, c_d) = \vec{b} \cdot L^{(i)}, \tag{47}$$

with $L^{(i)}$ denoting the i-th column of L. To solve the problem in (46), let us define the Lagrangian function

$$\mathcal{L}(k_1, \dots, k_d, \lambda) = \sum_{i=1}^{d} c_i k_i + \lambda\left(\sum_{i=1}^{d} k_i^2 - 1\right). \tag{48}$$

Differentiating (48) with respect to k_i leads to

$$k_i = -\frac{c_i}{2\lambda}, \quad i = 1, \dots, d. \tag{49}$$

Plugging (49) into the constraint in (46), we get $\frac{1}{4\lambda^2}\sum_{i=1}^{d} c_i^2 = 1$, which gives $\lambda = \frac{\sqrt{\sum_{i=1}^{d} c_i^2}}{2}$. From (49),

$k_i = -\frac{c_i}{\sqrt{\sum_{i=1}^{d} c_i^2}}$. Thus, considering (47), we can write

$$\vec{k} = -\frac{L^\intercal \vec{b}}{\sqrt{\vec{c}\,\vec{c}^\intercal}} = -\frac{L^\intercal \vec{b}}{\sqrt{\vec{b}^\intercal\, L L^\intercal \vec{b}}} = -\frac{L^\intercal \vec{\sigma}}{\sqrt{\vec{\sigma}^\intercal\, L L^\intercal \vec{\sigma}}},$$

where in the third equality we used the definition of $b_i, i = 1, \ldots, d$. Next, note that the denominator can also be written as $\sqrt{\vec{\sigma}^\intercal\, C \vec{\sigma}}$, which corresponds to the standard deviation of $\sum_{i=1}^{d} X_i$. Thus, we finally obtain $\vec{a}^\intercal = L\vec{k} = -\frac{C\vec{\sigma}}{\sqrt{\vec{\sigma}^\intercal\, C\, \vec{\sigma}}}$, which is our solution. □