Tontines with mixed cohorts: actuarial fairness and social welfare

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A note on financial fairness in tontines with mixed cohorts

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Abstract

We study fairness in tontines with heterogeneous cohorts, distinguishing between collective and individual fairness. For a given tontine, we show that collective fairness is always achievable and that it widely implies individual fairness. Furthermore, we study a utilitarian social planner setting up a tontine on behalf of the heterogeneous collective and analyze the expected utility of pensioners. In particular, we compare the resulting collectively optimal tontine to existing schemes in the literature and separate single-cohort tontines. We find that a tontine constructed by a social planner can outperform existing tontine schemes as it is able to better reflect individual risk preferences.

Keywords: Tontine, optimal retirement product design, individual fairness, collective fairness

JEL: G22, G52, H31, J32

1 Introduction

Societal challenges have generated a growing stream of literature on innovative retirement products providing random, mortality-linked benefits to pensioners. Such products are known as pooled annuity funds, group-self annuitization schemes and tontines (cf. Piggott et al. (2005), Sabin (2010), Donnelly et al. (2014), Milevsky and Salisbury (2015)). Although they carry different names, all these products have the similarity that a pool of pensioners shares mortality risks. Given a large enough pool, the idiosyncratic mortality risk can be diversified, while the pool shares the systematic mortality risk. Among the main challenges of these products lies the fact that pensioners with different age and wealth typically cannot be simply joint in one and the same scheme without discriminating at least some pensioners. While the majority of the literature in this field assumes a homogeneous pool for analytical convenience (e.g. Milevsky and Salisbury (2015), Chen et al. (2019, 2020)), there is also some literature dealing with heterogeneous cohorts (e.g. Sabin (2010), Donnelly et al. (2014), Milevsky and Salisbury (2016), and Chen et al. (2021c)). Milevsky and Salisbury (2016) come up with the concept *equitability*, a weaker concept than fairness, which states that each individual loses an identical fraction of their wealth if participating in the tontine with mixed cohorts. Chen et al. (2021c) determine the optimal payment design for a given policyholder in a tontine with mixed cohort, and do not find a unique withdrawal rate which satisfies the fairness condition simultaneously for all the individuals. In this article, we show that individual fairness is achievable in a closed scheme with heterogeneous cohorts as introduced in Milevsky and Salisbury (2016) and extend their article by considering a tontine scheme generated by a social planner which is optimal for the *collective* of pensioners with potentially different utility preferences.

Actuarially fair annuities are typically considered to be the optimal source of retirement income (Yaari (1965)). However, the inclusion of safety loadings in annuity premiums opens a market for mortality risk-carrying products like tontines, which contain lower safety loadings than annuities because providers do not promise guaranteed benefits (Milevsky and Salisbury (2015), Chen et al. (2020)). In fact, given a sufficiently large pool of policyholders, safety loadings can be nearly neglected in tontines entirely (Chen et al. (2019)). Thus, if tontines can be designed fairly for each participant, they could be offered at (almost) actuarially fair prices, a huge advantage for any retiree who is not fond of full annuitization.[1] We therefore believe that it is important to extend the article

[1]Naturally, there will still be some fees, but these will be small compared to annuities, as pointed

Milevsky and Salisbury (2016) to actuarially fair tontines for heterogeneous groups. In particular, we follow Milevsky and Salisbury (2016)'s tontine design and assume that individuals participate in the collective withdrawal rate by deterministic participation rates.

In this article, we distinguish between collective and individual fairness. In the latter one, we postulate that, for each policyholder, the expected present value of future benefits shall be equal to the single up-front premium paid by this policyholder. Collective fairness is a weaker requirement as it only requires the present value of future benefits that the pool receives to be equal to the sum of the initial up-front premiums. Hence, it is clear that individual fairness implies collective fairness. We extend the model setup in Milevsky and Salisbury (2016) by including systematic mortality risk and prove that the design they propose is not collectively fair, which prohibits it from being individually fair for all the individuals. Consequently, we propose an alternative way to design the withdrawal rate which ensures collectively fairness. Furthermore, we then derive conditions under which individual fairness is achievable and show that it is widely possible to achieve individual fairness.

In the second part of this paper, we analyze the expected discounted lifetime utility of individual tontine participants. While it is clearly not possible to achieve a withdrawal rate which optimizes the expected discounted lifetime utility of each cohort simultaneously (Chen et al. (2021c)), we are interested in the performance of a social planner maximizing the weighted sum of the individual utility functions to achieve a withdrawal rate which is at least optimal for the collective of heterogeneous policyholders. This type of collective utility function is frequently considered in the finance literature, see e.g. Wilson (1968), Dumas (1989), Weinbaum (2009) and Jensen and Nielsen (2016). The resource constraint of the social planner is the collective fairness criterion and we assume that the social planner chooses the *participation rates* in the mixed-cohort tontine in such a way that individual fairness is fulfilled. The concept of "participation rate" is taken from Milevsky and Salisbury (2016). It reflects the contract price for different policyholders, particularly with various ages and the relative survival benefit one could get out of a tontine contract. A lower participation rate implies a higher contract price for the policyholder. We then compare the resulting withdrawal rate to tontines with homogeneous cohorts as well as the proportional and natural tontine as introduced by Milevsky and Salisbury (2016). We find that all these tontine schemes deliver approximately identical certainty equivalents under logarithmic utility. Under heterogeneous

out in e.g. Chen et al. (2021a).

power utility preferences, however, we find that the social planner and the single-cohort tontines clearly outperform the proportional and natural tontine. The reason for this is that the natural and proportional tontine are unable to take into account the degrees of relative risk aversion of policyholders. In contrast to Milevsky and Salisbury (2016), we find that the benefits of pooling heterogeneous cohorts do not clearly exceed the benefits of individually utility-maximizing withdrawal rates in single-cohort tontines.

The remainder of this article is structured as follows: In Section 2, we consider a tontine with a given withdrawal rate and derive conditions under which such a tontine is collectively and individually fair. In Section 3, we then solve the collective optimization problem of the social planner and analyze the expected discounted lifetime utility of the heterogeneous policyholders. In Section 4, we conclude. Some proofs and technical details as well as a pseudo code are collected in the appendix.

2 Achieving fairness in given tontines

In this section, it is our goal to derive conditions under which fairness in mixed-cohort tontines can be achieved, if withdrawal rates are assumed to be given. In particular, it is not the goal of this section to derive individually utility-maximizing withdrawal rates as, for instance, done in Milevsky and Salisbury (2015) and Chen et al. (2019, 2021c).

2.1 Model setup

We consider L cohorts that differ in initial wealth, age and risk preferences. We denote the initial size of cohort $j \in \{1, \dots, L\}$ by n_i, the age of the members in cohort j by x_j and the initial wealth of a member of cohort j by w_j. We assume that the members in each cohort are identical copies of each other. The total initial pool size is then $n = n_1 + \cdots + n_L$.

The remaining lifetime of policyholder i will be denoted by T_i for $i = 1, \dots, n$. The (possibly stochastic) force of mortality of each member is μ_{x+t}, i.e. all the members are subject to the same mortality law (but still differ in their ages). We use $\mathcal{F}_t := \sigma(\{\mu_{x+s}\}_{s \leq t})$ to denote the sigma-algebra containing the information regarding the systematic mortality risk up to time t. Furthermore, we introduce the notation

$$S_x(t) := \mathbb{E}\left[\mathbb{1}_{\{T>t\}} \mid \mathcal{F}_t\right] = e^{-\int_0^t \mu_{x+s} \mathrm{d}s}$$

for the random survival probabilities conditional on the systematic mortality outcome. The (deterministic) survival probabilities are then given by

$$s_x(t) := \mathbb{E}\left[\mathbb{1}_{\{T>t\}}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{T>t\}} \mid \mathcal{F}_t\right]\right] = \mathbb{E}\left[e^{-\int_0^t \mu_{x+s}\mathrm{d}s}\right].$$

The number of policyholders alive at time t is then given by

$$N(t) = \sum_{j=1}^{L} N_j(t),$$

where $(N_j(t) \mid \mathcal{F}_t) \sim \mathrm{Bin}(n_j, S_{x_j}(t))$. Conditional on $\mathcal{F}_t$, the overall number of living policyholders $N(t)$ follows a Poisson Binomial distribution.

Following Milevsky and Salisbury (2016), the payoff to an individual policyholder is given by

$$b^{(i)}(t) := wd(t) \cdot \frac{\pi_i w_i}{\sum_{j=1}^{L} \pi_j w_j N_j(t)} \mathbb{1}_{\{T_i>t\}} \tag{1}$$

where $d(t)$ is a deterministic withdrawal rate specified at the beginning of the contract and $w = \sum_{i=1}^{L} n_i w_i$ is the total initial wealth from all the cohorts. π_i is the so-called participation rate, or, in other words, $1/\pi_i$ can be interpreted as the share price for an individual i to participate in the tontine product. The main purpose of this parameter is to arrive at a higher level of fairness among heterogeneous policyholders, particularly those with different ages. Thinking of policyholders with different ages, with all the other parameters being identical, roughly speaking, the older shall be entitled to higher tontine payments, as the entire period during which they obtain payoffs is expected to be shorter. In this sense, the participation level for older policyholders shall be higher, or equivalently, the share price for older policyholders shall be lower. The quantity $\pi_i w_i$ can be considered as the number of shares of individual i.

In the remainder of this section, we will assume that $d(t)$ is a given withdrawal rate and analyze under which conditions fair participation rates π_i exist and whether they are unique.

2.2 Collective and individual fairness

In this article, we disregard financial market risk to focus exclusively on mortality risk. Let r be the risk-free interest rate. The actuarial premium for an individual j in a given cohort can be computed as the expected present value of the benefits:

$$\begin{aligned} P_0^j &= \mathbb{E}\left[\int_0^\infty e^{-rt} w d(t) \frac{\pi_j w_j}{\sum_{i=1}^L \pi_i w_i N_i(t)} \mathbb{1}_{\{T_j > t\}} \mathrm{d}t\right] \\ &= \int_0^\infty e^{-rt} \mathbb{E}\left[w d(t) \frac{\pi_j w_j}{\sum_{i=1}^L \pi_i w_i N_i(t)} \mathbb{1}_{\{T_j > t\}}\right] \mathrm{d}t \\ &= \int_0^\infty e^{-rt} w d(t) \mathbb{E}\left[S_{x_j}(t) \mathbb{E}\left[\frac{\pi_j w_j}{\sum_{i=1}^L \pi_i w_i N_i(t)} \;\middle|\; \mathcal{F}_t, T_j > t\right]\right] \mathrm{d}t. \end{aligned} \tag{2}$$

Using (2), we can now specify the term "fairness".

Definition 2.1. *We distinguish between collective and individual fairness:*

- *Individual fairness is defined for a member in cohort* $j \in \{1, \dots, L\}$ *if* $w_j = P_0^j$.
- *Collective fairness is defined as*

$$w = \sum_{i=1}^n P_0^i = \sum_{j=1}^L n_j P_0^j.$$

If individual fairness holds for all the individuals in the mixed cohorts, it implies the collective fairness. However, the reverse does not hold through. Milevsky and Salisbury (2016) have shown that their design is not collectively fair (and thus cannot be individually fair for all policyholders). For the sake of completeness, we briefly want to show that their design is not collectively fair in our slightly generalized setting and, particularly, discuss the main assumption responsible for this result.

Proposition 2.2. *Following Milevsky and Salisbury (2016), i.e. particularly assuming that*

$$\int_0^\infty e^{-rt} d(t) \mathrm{d}t = 1,$$

the mixed-cohort payoff design in (1) *is not collectively fair.*

Proof. First, note that the premium charged from the collective, or in other words, the sum of the individual premiums, is given by

$$\sum_{j=1}^n P_0^j = \sum_{j=1}^n \int_0^\infty e^{-rt} \mathbb{E}\left[w d(t) \frac{\pi_j w_j}{\sum_{i=1}^L \pi_i w_i N_i(t)} \mathbb{1}_{\{T_j > t\}}\right] \mathrm{d}t$$

$$= \int_0^\infty e^{-rt} w d(t) \sum_{j=1}^n \mathbb{E}\left[\frac{\pi_j w_j}{\sum_{i=1}^L \pi_i w_i N_i(t)} \mathbb{1}_{\{T_j > t\}} \right] \mathrm{d}t.$$

Here, it holds

$$\begin{aligned}
&\sum_{j=1}^n \mathbb{E}\left[\frac{\pi_j w_j}{\sum_{i=1}^L \pi_i w_i N_i(t)} \mathbb{1}_{\{T_j > t\}} \right] = \mathbb{E}\left[\sum_{j=1}^n \frac{\pi_j w_j}{\sum_{i=1}^L \pi_i w_i N_i(t)} \mathbb{1}_{\{T_j > t\}} \right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\sum_{j=1}^n \frac{\pi_j w_j}{\sum_{i=1}^L \pi_i w_i N_i(t)} \mathbb{1}_{\{T_j > t\}} \;\middle|\; \mathcal{F}_t \right] \right] \\
&= \mathbb{E}\left[P(N(t) > 0 \mid \mathcal{F}_t) \mathbb{E}\left[\sum_{j=1}^n \frac{\pi_j w_j}{\sum_{i=1}^L \pi_i w_i N_i(t)} \mathbb{1}_{\{T_j > t\}} \;\middle|\; \mathcal{F}_t, N(t) > 0 \right] \right] + \mathbb{E}\left[P(N(t) = 0 \mid \mathcal{F}_t) \cdot 0 \right] \\
&= \mathbb{E}\left[P(N(t) > 0 \mid \mathcal{F}_t) \right] \\
&= \mathbb{E}\left[1 - \prod_{j=1}^n (1 - S_{x_j}(t)) \right],
\end{aligned}$$

because

$$\sum_{j=1}^n \pi_j w_j \mathbb{1}_{\{T_j > t\}} = \sum_{i=1}^L \pi_i w_i N_i(t),$$

and π_j and w_j are identical within the cohorts. It has then the consequence

$$\begin{aligned}
\sum_{j=1}^n P_0^j &= \int_0^\infty e^{-rt} w d(t) \mathbb{E}\left[1 - \prod_{j=1}^n (1 - S_{x_j}(t)) \right] \mathrm{d}t \\
&< \int_0^\infty e^{-rt} w d(t) \mathrm{d}t = w. \qquad (3)
\end{aligned}$$

The inequality in (3) shows that the fairness does not hold in the mixed-cohort tontine on the collective level. □

Milevsky and Salisbury (2016) assume that the withdrawal rate $d(t)$ is chosen in such a way that the tontine provider makes payments up to an infinite time horizon although no living policyholders are left, or, in other words, they assume that the pool contains at least one policyholder who lives forever. Since the tontine pool size considered here is finite and since it is not possible for a policyholder to live forever, we slightly adjust the constraint on the withdrawal rate $d(t)$. For this, we introduce the notation

$$\begin{aligned} P_0 &:= \mathbb{E}\left[\int_0^\infty e^{-rt} d(t) \mathbb{1}_{\{N(t)>0\}} \mathrm{d}t\right] \\ &= \int_0^\infty e^{-rt} d(t) \mathbb{E}\left[1 - \prod_{j=1}^{n} (1 - S_{x_j}(t))\right] \mathrm{d}t, \end{aligned}$$

i.e. wP_0 denotes the present value of payments that the tontine provider pays out to the policyholders.

Proposition 2.3. *If* $P_0 = 1$, *the mixed-cohort payoff design in* (1) *is collectively fair.*

Proof. It is a straightforward calculation (see the proof of Proposition 2.2) to show that

$$\sum_{j=1}^{n} P_0^j = w \cdot P_0.$$

Hence, assuming $P_0 = 1$ directly leads to the collective fairness. □

Given that the collective fairness is fulfilled, it seems natural to examine whether there are some choices of $\pi_1, \ldots, \pi_L$ such that the individual fairness is additionally fulfilled. For the case with two cohorts $L = 2$, we can theoretically show that individual fairness for all individuals can be ensured under mild assumptions.

Proposition 2.4. *Let* $L = 2$ *with* $n_1 > 0$, $n_2 > 0$, $w_1 > 0$, $w_2 > 0$, *assume that the collective fairness is fulfilled, i.e.*

$$P_0 = 1, \tag{4}$$

and that

$$n_1 w_1 < n_2 w_2 \left(\frac{\int_0^\infty e^{-rt} d(t) \mathbb{E}\left[1 - (1 - S_{x_1}(t))^{n_1}\right] \mathrm{d}t}{1 - \int_0^\infty e^{-rt} d(t) \mathbb{E}\left[1 - (1 - S_{x_1}(t))^{n_1}\right] \mathrm{d}t} \right). \tag{5}$$

Further, assume without loss of generality that $\pi_1 = 1$. *Then, there exists a unique value* π_2 *such that the individual fairness is satisfied for both cohorts.*

Proof. We want to have

$$w_i = P_0^i, \quad i = 1, 2. \tag{6}$$

Now we set $\pi_1 := 1$ without loss of generality. Under this specification, we get:

$$P_0^1 = \int_0^\infty e^{-rt}\mathbb{E}\left[wd(t)\frac{w_1}{w_1N_1(t)+w_2\pi_2N_2(t)}\mathbb{1}_{\{T_1>t\}}\right]\mathrm{d}t,$$
$$P_0^2 = \int_0^\infty e^{-rt}\mathbb{E}\left[wd(t)\frac{\pi_2 w_2}{w_1N_1(t)+w_2\pi_2N_2(t)}\mathbb{1}_{\{T_2>t\}}\right]\mathrm{d}t.$$

Note that for (6) to be fulfilled, it suffices to find π_2 such that $w_1 = P_0^1$. Once this is fulfilled, the second condition $w_2 = P_0^2$ follows directly from the collective fairness condition (4). It is clear that $P_0^1 : [0,\infty) \to \left(0, \int_0^\infty e^{-rt}\mathbb{E}\left[\frac{wd(t)}{N_1(t)}\mathbb{1}_{\{T_1>t\}}\right]\mathrm{d}t\right]$, $\pi_2 \mapsto P_0^1(\pi_2)$ is a strictly decreasing and thus bijective function in π_2. Therefore, if

$$w_1 < \int_0^\infty e^{-rt}\mathbb{E}\left[\frac{wd(t)}{N_1(t)}\mathbb{1}_{\{T_1>t\}}\right]\mathrm{d}t, \tag{7}$$

it follows that there exists a unique positive number π_2^* such that $P_0^1(\pi_2^*) = w_1$.

Note that

$$\begin{aligned}
\int_0^\infty e^{-rt}\mathbb{E}\left[\frac{wd(t)}{N_1(t)}\mathbb{1}_{\{T_1>t\}}\right]\mathrm{d}t &= \int_0^\infty e^{-rt}\frac{wd(t)}{n_1}\mathbb{E}\left[\frac{n_1}{N_1(t)}\mathbb{1}_{\{T_1>t\}}\right]\mathrm{d}t \\
&= \int_0^\infty e^{-rt}\frac{wd(t)}{n_1}\mathbb{E}\left[1-(1-S_{x_1}(t))^{n_1}\right]\mathrm{d}t \\
&= \int_0^\infty e^{-rt}\left(w_1+\frac{n_2w_2}{n_1}\right)d(t)\mathbb{E}\left[1-(1-S_{x_1}(t))^{n_1}\right]\mathrm{d}t \\
&= w_1\int_0^\infty e^{-rt}d(t)\mathbb{E}\left[1-(1-S_{x_1}(t))^{n_1}\right]\mathrm{d}t \\
&\quad + \frac{n_2w_2}{n_1}\int_0^\infty e^{-rt}d(t)\mathbb{E}\left[1-(1-S_{x_1}(t))^{n_1}\right]\mathrm{d}t.
\end{aligned}$$

With this, we can reformulate (7) to (5). □

Note that for larger values of n_1, the denominator in (5) will be close to zero, because $\int_0^\infty e^{-rt}d(t)\mathbb{E}\left[1-(1-S_{x_1}(t))^{n_1}\right]\mathrm{d}t$ is close to 1. Thus, in realistic situations with appropriate pool sizes, individual fairness is achievable.

Next, we prove that for any cohort size, given the existence of a set of fair participation rates, this set is unique up to multiplicative constant. For this, we directly follow a proof provided in Milevsky and Salisbury (2016).

Proposition 2.5. *Let $L \geq 2$ and $d(t)$ be given with $n_i > 0$ and $w_i > 0$ and assume that a set of fair participation rates $\pi = (\pi_1, \ldots, \pi_L)$ with $\pi_j \in (0,\infty)$ exists. Then,* *this set is unique up to a multiplicative constant.*

Proof. The proof follows similar steps as the proof of Theorem 4(a) in Milevsky and Salisbury (2016) and is provided in Appendix A.1 for the sake of completeness. □

The questions is now whether a set of fair participation rates exists for larger numbers of cohorts $L \geq 3$. For such a number of cohorts, we get:

$$w_1 = \int_0^\infty e^{-rt}\mathbb{E}\left[wd(t)\frac{w_1}{w_1N_1(t) + \sum_{j=2}^L w_j\pi_jN_j(t)}\mathbb{1}_{\{T_1>t\}}\right]\mathrm{d}t,$$
$$w_i = \int_0^\infty e^{-rt}\mathbb{E}\left[wd(t)\frac{\pi_i w_i}{w_1N_1(t) + \sum_{j=2}^L w_j\pi_jN_j(t)}\mathbb{1}_{\{T_i>t\}}\right]\mathrm{d}t, \ \ i = 2, \dots, L-1.$$

Note that we can again omit the last equation ($i = L$) due to the collective fairness. Therefore, we have $L-1$ nonlinear equations and $L-1$ unknowns $(\pi_2, \dots, \pi_L)$. In the following, we consider the expected present values of future benefits P_0^i as functions of the participation rates. Note that these functions

$$P_0^1 : (0,\infty) \to \left(0, \int_0^\infty e^{-rt}\mathbb{E}\left[wd(t)\frac{w_1}{w_1N_1(t) + \sum_{j=3}^L w_j\pi_jN_j(t)}\mathbb{1}_{\{T_i>t\}}\right]\mathrm{d}t\right),$$
$$\pi_2 \mapsto P_0^1(\pi_2),$$
$$P_0^i : (0,\infty) \to \left(0, \int_0^\infty e^{-rt}\mathbb{E}\left[wd(t)\frac{\pi_i w_i}{w_1N_1(t) + \sum_{j=2, j\neq i+1}^L w_j\pi_jN_j(t)}\mathbb{1}_{\{T_i>t\}}\right]\mathrm{d}t\right),$$
$$\pi_{i+1} \mapsto P_0^i(\pi_{i+1}), \ \ i = 2, \dots, L-1$$

are strictly decreasing in π_2 and π_{i+1}, treating π_j, $j = 3, \dots, L$ and π_j, $j \neq i+1$ as given constants, respectively. In other words, for any values of π_j, $j = 3, \dots, L$, fulfilling

$$w_1 < \int_0^\infty e^{-rt}\mathbb{E}\left[wd(t)\frac{w_1}{w_1N_1(t) + \sum_{j=3}^L w_j\pi_jN_j(t)}\mathbb{1}_{\{T_i>t\}}\right]\mathrm{d}t, \tag{8}$$

we can find a unique value π_2^* such that $P_0^1(\pi_2^*) = w_1$. Similarly, for any values π_j, $j \neq i+1$ satisfying

$$w_i < \int_0^\infty e^{-rt}\mathbb{E}\left[wd(t)\frac{\pi_i w_i}{w_1N_1(t) + \sum_{j=2, j\neq i+1}^L w_j\pi_jN_j(t)}\mathbb{1}_{\{T_i>t\}}\right]\mathrm{d}t, \tag{9}$$

we can find a unique value π_{i+1}^* such that $P_0^i(\pi_{i+1}^*) = w_i$. Thus, we can successively determine all the values $\pi_2^*, \ldots, \pi_L^*$ until all the fairness conditions are met. This can, for instance, be done using the numerical procedure provided in Appendix B.

2.3 Numerical example

Throughout the numerical analyses, we rely on a shocked Gompertz law. The deterministic Gompertz law (see Gompertz (1825)) is frequently used in actuarial science, particularly for retirement planning (see e.g. Milevsky (2020)). In this paper, we follow e.g. Lin and Cox (2005) and apply a stochastic shock to this deterministic mortality law to take account of the systematic mortality risk. Such a model can be motivated by the fact that regulators in many countries require insurers to test their balance sheet against various stress scenarios. In total, the force of mortality is, for any x and $t \geq 0$, given by

$$\mu_{x+t} = (1-\epsilon)\frac{1}{b}e^{\frac{x+t-m}{b}},$$

where $m > 0$ denotes the modal age at death, $b > 0$ is the dispersion coefficient and ϵ is a random shock taking values in $(-\infty, 1)$. Table 1 provides the base case parameters used in the subsequent numerical analyses.

Cohorts $L = 3$	Cohort sizes $n_i = 500$	Initial wealth levels $w_1 = 100$, $w_2 = 200$, $w_3 = 300$
Risk-free rate $r = 0.01$	Participation rate $\pi_1 = 1$	Initial ages $x_1 = 65$, $x_2 = 70$, $x_3 = 75$
Modal age $m = 88.721$	Dispersion $b = 10$	Longevity shock $\epsilon \sim \mathcal{N}_{(-\infty,1)}(-0.0035, 0.0814^2)$

Table 1: Base case parameter setup. The index i ranges from 1 to L.

The parameters are chosen due to the following reasons:

- For the overall pool size, we follow Qiao and Sherris (2013) who recommend a pool size of at least 1000 for modern tontines.

- Based on the ongoing low interest rate environment across most countries, we set the constant risk-free interest rate close to zero. For example, the German average risk-free rate of investment in 2019 was 1.1% (see Statista (2019)).

- The ages of the retirees 65, 70 and 75 are typical retirement ages. The initial wealth levels of the retirees increase in the age, since individuals who decide to postpone their retirement have more years to earn income than individuals who retire at an earlier age.

- For the values of m and b, we follow Milevsky and Salisbury (2015).

- Concerning the longevity shock ϵ, we follow Chen et al. (2019) and assume that it follows a truncated normal distribution on the interval $(-\infty, 1)$. The parameters used for this distribution are also taken from Chen et al. (2019).

As a simple example, we consider a flat tontine as defined in Milevsky and Salisbury (2015), i.e. we consider a constant withdrawal rate

$$d(t) = d = \frac{1}{\int_0^\infty e^{-rt}\mathbb{E}\left[1 - \prod_{j=1}^n (1 - S_{x_j}(t))\right] \mathrm{d}t}.$$

In Table 2, we present the resulting fair participation rates.

$w_i = 100i$		
	$n_i = 100$	$n_i = 500$
$x = 70$	$\pi_2 = 5.49$	$\pi_2 = 6.57$
$x = 75$	$\pi_3 = 11.76$	$\pi_3 = 15.16$
$w_i = 100$		
	$n_i = 100$	$n_i = 500$
$x = 70$	$\pi_2 = 2.52$	$\pi_2 = 2.93$
$x = 75$	$\pi_3 = 4.42$	$\pi_3 = 5.36$

Table 2: Fair participation rates π_2 and π_3 for the flat tontine. We rely on the parameters introduced in Table 1.

Naturally, the participation rates increase with age, as older individuals are entitled to higher retirement benefits. Furthermore, we observe that the participation rates increase more steeply in the age if initial wealth levels are increasing in age as well. This is another natural result, since individuals who make higher contributions are, by the individual fairness, entitled to even higher benefits. Finally, we observe that an increase in the pool size leads to a rise in the participation rate, but does not affect the relation between the participation rates.

3 Optimal tontine under fairness

3.1 Social planner's problem

It is clear that the withdrawal rate $d(t)$ of the mixed-cohort tontine considered in this article cannot be determined in such a way that the expected discounted lifetime utilities of all cohorts are maximized, as it is frequently done for homogeneous-cohort tontines (cf. Milevsky and Salisbury (2015) and Chen et al. (2019, 2020)). In the following, we therefore choose a withdrawal rate $d(t)$ which maximizes a weighted sum of the individual utility functions. In this sense, the resulting withdrawal rate $d(t)$ is optimal for the collective, but not optimal for a single cohort. In particular, this approach is particularly different from the approach taken in Milevsky and Salisbury (2016), where the natural tontine design introduced in Milevsky and Salisbury (2015) is generalized to heterogeneous cohorts.

Let U_i be the utility function and ρ_i be the subjective discount factor of individual i. Furthermore, let $\beta_1, \ldots, \beta_n$ be nonnegative numbers adding up to 1. Then, the optimization problem of the social planner is given by

$$\begin{aligned} &\max_{d(t)} \mathbb{E}\left[\int_0^\infty \sum_{i=1}^n e^{-\rho_i t}\beta_i U_i\left(wd(t)\cdot\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\right)\mathbb{1}_{\{T_i>t\}}\mathrm{d}t\right] \\ &\text{subject to } \sum_{j=1}^n P_0^j = \int_0^\infty e^{-rt} wd(t)\mathbb{E}\left[1-\prod_{j=1}^n (1-S_{x_j}(t))\right]\mathrm{d}t = w, \end{aligned} \tag{10}$$

where w is the total initial wealth. If all the individual utility preferences are identical and given by a log-utility function, we can find an explicit solution to this optimization problem.

Theorem 3.1. *Assume that $U_i(\cdot) = \ln(\cdot)$ for all $i = 1, \ldots, n$. Then, the optimization problem* (10) *has the following explicit solution:*

$$d^*(t) = \frac{\sum_{i=1}^n e^{-\rho_i t}\beta_i \mathbb{E}\left[\mathbb{1}_{\{T_i>t\}}\right]}{\lambda e^{-rt}\mathbb{E}\left[1-\prod_{j=1}^n (1-S_{x_j}(t))\right]}, \tag{11}$$

where the Lagrangian multiplier is given by

$$\lambda = \sum_{i=1}^n \beta_i \int_0^\infty e^{-\rho_i t}\mathbb{E}\left[\mathbb{1}_{\{T_i>t\}}\right]\mathrm{d}t.$$

Proof. See Appendix A.2. □

Note that the optimal withdrawal rate (11) depends on individual discount factors, utility weights and future lifetimes, but not on the individual participation rates. We can therefore treat this withdrawal rate as given and proceed as described in Section 2 to determine the individually fair participation rates.

If the utility functions are not identical, the expected discounted lifetime utility can be rewritten in the following way:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty \sum_{i=1}^n e^{-\rho_i t}\beta_i U_i\left(wd(t)\cdot\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\right)\mathbb{1}_{\{T_i>t\}}\mathrm{d}t\right]\\
&=\int_0^\infty \sum_{i=1}^n \mathbb{E}\left[e^{-\rho_i t}\beta_i U_i\left(wd(t)\cdot\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\right)\mathbb{1}_{\{T_i>t\}}\right]\mathrm{d}t\\
&=\int_0^\infty \sum_{i=1}^n \mathbb{E}\left[S_{x_i}(t)\mathbb{E}\left[e^{-\rho_i t}\beta_i U_i\left(wd(t)\cdot\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\right)\,\middle|\,\mathcal{F}_t, T_i>t\right]\right]\mathrm{d}t\\
&=\int_0^\infty \sum_{i=1}^L n_i\mathbb{E}\left[S_{x_i}(t)\mathbb{E}\left[e^{-\rho_i t}\beta_i U_i\left(wd(t)\cdot\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\right)\,\middle|\,\mathcal{F}_t, T_i>t\right]\right]\mathrm{d}t,
\end{aligned}$$

where we have used that the individuals within cohort i are all identical copies of each other, i.e. the expectations are identical and can be multiplied by n_i. The Lagrangian is given by

$$\begin{aligned}
\mathcal{L}=&\int_0^\infty \sum_{i=1}^L \mathbb{E}\left[S_{x_i}(t)\mathbb{E}\left[e^{-\rho_i t}n_i\beta_i U_i\left(wd(t)\cdot\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\right)\,\middle|\,\mathcal{F}_t, T_i>t\right]\right]\mathrm{d}t\\
&+\lambda\left(w-\int_0^\infty e^{-rt}wd(t)\mathbb{E}\left[1-\prod_{j=1}^n(1-S_{x_j}(t))\right]\mathrm{d}t\right).
\end{aligned}$$

The first-order condition is then given by

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{E}\left[S_{x_i}(t)\mathbb{E}\left[e^{-\rho_i t}\beta_i U_i'\left(wd(t)\cdot\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\right)w\cdot\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\,\middle|\,\mathcal{F}_t, T_i>t\right]\right]\\
&=\lambda e^{-rt}w\mathbb{E}\left[1-\prod_{j=1}^n(1-S_{x_j}(t))\right].
\end{aligned}$$

To solve this equation, we need to rely on numerical procedures. In particular, we need to simultaneously determine the optimal withdrawal rate $d(t)$ and the individually fair participation rates π_i.

For our numerical analyses, we assume the utility functions to of the constant relative risk aversion (CRRA) type, i.e. for $\gamma_i > 0$, the utility function U_i is defined by

$$U_i(y) = \begin{cases} \frac{y^{1-\gamma_i}}{1-\gamma_i}, & \gamma_i \neq 1 \\ \ln(y), & \gamma_i = 1. \end{cases}$$

In the utility function $\sum_{i=1}^{n} \beta_i U_i(\cdot)$, we add different types of CRRA utility functions U_i. To make sure that the units in the collective utility function can be added, we need to ensure that the weights β_i are chosen in such a way that $\beta_i U_i$ owns the same unit for all i. This is e.g. ensured by the choice of weights

$$\beta_i = \frac{\bar{w}^{\gamma_i}}{\sum_{j=1}^{L} n_j \bar{w}^{\gamma_j}}, \quad \bar{w} = \frac{1}{n} \sum_{i=1}^{L} n_i w_i,$$

which we have taken from Chen et al. (2021b).

In this section, we fix the first two cohorts introduced in Table 1 as base case parameters. This allows us to analyze wealth and utility transfers between two cohorts with differential mortality in a similar way as, for instance, Bommier et al. (2011). Throughout the numerical analyses, we assume $\rho_i = r$ to ensure a fair comparison between the tontine of the social planner and Milevsky and Salisbury (2016)'s designs. For the CRRA utility, we assume that $\gamma_1 = 6$ and $\gamma_2 = 8$, i.e. risk aversion increases with age, as found in the literature (at least from age 65 on, see e.g. Riley Jr and Chow (1992) and DaSilva et al. (2019)).

In Figure 1, we show the optimal withdrawal rate $d(t)$ obtained under log utility and power utility. Note that the optimal withdrawal rate under log utility is independent of π_1 and π_2. For the power utility, we set $\pi_1 = 1$ and $\pi_2 = 1.222$ to achieve individual fairness. We observe that the optimal withdrawal rate $d^*(t)$ is decreasing and therefore coincides roughly with optimal withdrawal rates derived in the literature (cf. Milevsky and Salisbury (2015)). Furthermore, we observe that higher risk aversions lead individuals to prefer a flatter withdrawal rate.

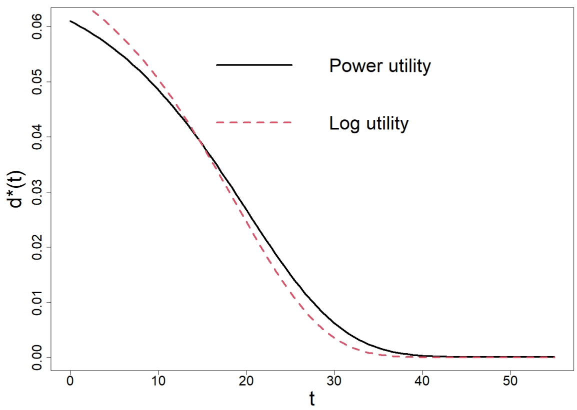

Figure 1: Optimal withdrawal rate $d^*(t)$ depending on time t for log utility and power utility with $\gamma_1 = 6$ and $\gamma_2 = 8$. The parameters are chosen as in Table 1. Furthermore, we assume $\rho_i = r$.

3.2 Certainty equivalents and additional designs

We compare the social planner's tontine to tontines with homogeneous cohorts set up for each of the separate cohorts. Note that for such homogeneous-cohort-tontines, an explicit solution to the optimal withdrawal rate exists under CRRA utility preferences (see e.g. Chen et al. (2019)). In addition, we take two designs introduced in Milevsky and Salisbury (2016) into consideration. First, the *proportional* tontine is specified by

$$d(t) = \sum_{j=1}^{L} \frac{n_j w_j}{w} \cdot \frac{s_{x_j}(t)}{\bar{a}_{x_j}},$$

where $\bar{a}_{x_j} = \int_0^\infty e^{-rt} s_{x_j}(t) \mathrm{d}t$ is the money's worth of an annuity paying out 1 continuously until death. Note that this design is not collectively fair, since

$$\int_0^\infty e^{-rt} \mathbb{E}\left[1 - \prod_{j=1}^{n} (1 - S_{x_j}(t))\right] d(t) \mathrm{d}t < \int_0^\infty e^{-rt} d(t) \mathrm{d}t = 1.$$

If we want to use this tontine structure in our setting, we can easily modify it to the following collectively fair scheme:

$$d(t) = \frac{1}{\mathbb{E}\left[1 - \prod_{j=1}^{n} (1 - S_{x_j}(t))\right]} \sum_{j=1}^{L} \frac{n_j w_j}{w} \cdot \frac{s_{x_j}(t)}{\bar{a}_{x_j}}.$$

In a final step, we can then choose the participation rates π_j in such a way that individual fairness is achieved.

Furthermore, Milevsky and Salisbury (2016) consider the *natural* mixed-cohort tontine, specified by

$$d(t) = \sum_{j=1}^{L} \frac{\pi_j n_j w_j}{\sum_{i=1}^{L} \bar{a}_{x_i} \pi_i n_i w_i} \cdot s_{x_j}(t).$$

where the weights π_j can be chosen arbitrarily. Note that this design is not collectively fair either, since

$$\int_0^\infty e^{-rt} \mathbb{E}\left[1 - \prod_{j=1}^{n}(1 - S_{x_j}(t))\right] d(t)\mathrm{d}t < \int_0^\infty e^{-rt} d(t)\mathrm{d}t = 1.$$

Again, we can easily modify this withdrawal rate to the following collectively fair scheme:

$$d(t) = \frac{1}{\mathbb{E}\left[1 - \prod_{j=1}^{n}(1 - S_{x_j}(t))\right]} \sum_{j=1}^{L} \frac{\pi_j n_j w_j}{\sum_{i=1}^{L} \bar{a}_{x_i} \pi_i n_i w_i} \cdot s_{x_j}(t),$$

in which we can again choose the participation rates in such a way that individual fairness is achieved.

We analyze the utility benefit/loss generated by the collective problem by comparing it to optimal solutions when they are treated as separated cohorts. To compare the benefits resulting from the different tontines, we consider the certainty equivalent CE as the level of the deterministic annuity payoff that yields the same expected utility as a given mixed-cohort tontine. That is, the certainty equivalent CE_i of individual i is determined by

$$\int_0^\infty e^{-\rho_i t} s_{x_i}(t) U_i(\mathrm{CE}_i) = \mathbb{E}\left[\int_0^\infty e^{-\rho_i t} U_i \left(w d(t) \cdot \frac{\pi_i w_i}{\sum_{j=1}^{L} \pi_j w_j N_j(t)} \right) \mathbb{1}_{\{T_i > t\}} \mathrm{d}t \right].$$

Under CRRA utility functions, the certainty equivalent can be determined as

$$\mathrm{CE}_i = \begin{cases} \left((1-\gamma_i) U_i\left(\{\chi(t)\}_{t\geq 0}\right) \left(\int_0^\infty e^{-\rho_i t} s_{x_i}(t)\,\mathrm{d}t\right)^{-1} \right)^{\frac{1}{1-\gamma_i}}, & \gamma_i \neq 1 \\ e^{U_i\left(\{\chi(t)\}_{t\geq 0}\right)\left(\int_0^\infty e^{-\rho_i t} s_{x_i}(t)\,\mathrm{d}t\right)^{-1}}, & \gamma_i = 1, \end{cases}$$

depending on whether power or log utility is used.

3.3 Numerical results

In Table 3 we provide the certainty equivalents of the the mixed-cohort and the single-cohort tontines for log-utility. We observe that the proportional and natural tontine

	$x = 65$	$x = 70$	π_2
$n_1 = n_2 = 100$			
Homogeneous tontine	5.423	13.09	-
Social planner	5.40	13.12	1.236
Proportional tontine	5.417	13.12	1.216
Natural tontine	5.416	13.13	1.216
$n_1 = n_2 = 500$			
Homogeneous tontine	5.44	13.127	-
Social planner	5.42	13.130	1.230
Proportional tontine	5.43	13.14	1.211
Natural tontine	5.43	13.14	1.211

Table 3: Certainty equivalents obtained under log-utility. We rely on the parameters introduced in Table 1 and the participation rates making the heterogeneous tontine individually fair. Furthermore, we assume $\rho_i = r$.

slightly outperform the social planner for both pool sizes considered. However, they are unable to outperform the homogeneous tontines. In particular, we observe that the younger cohort prefers the homogeneous tontine to the mixed-cohort tontines. This result is reversed for the older cohort. All in all, the differences between the different tontine designs are rather small. Furthermore, we observe that a rise in the pool size increases all certainty equivalents, a natural result which is consistent with the literature.

In Table 4, we provide the certainty equivalents of the the mixed-cohort and the single-cohort tontines for CRRA utility. We observe that the social planner and the homogeneous tontines now outperform the natural and proportional tontine. The reason for this result is that both these tontine designs are generalizations of the natural tontine introduced in Milevsky and Salisbury (2015) which is optimal for a log-utility maximizer. Furthermore, we observe again that the younger cohort prefers the homogeneous tontine to the mixed-cohort tontine set up by the social planner. This result is reversed for the older cohort. It is therefore not clear which approach of the two should be preferred and, therefore, one might as well argue that the decision to set up a single tontine with heterogeneous cohorts or separate tontines with homogeneous cohorts could be left with the provider of the plans.

	$x = 65$	$x = 70$	π_2
$n_1 = n_2 = 100$			
Homogeneous tontine	5.31	12.64	-
Social planner	5.27	12.89	1.236
Proportional tontine	5.16	12.00	1.216
Natural tontine	5.18	12.15	1.216
$n_1 = n_2 = 500$			
Homogeneous tontine	5.38	12.91	-
Social planner	5.36	13.02	1.222
Proportional tontine	5.28	12.68	1.211
Natural tontine	5.29	12.75	1.211

Table 4: Certainty equivalents obtained for $\gamma_1 = 6$ and $\gamma_2 = 8$. We rely on the parameters introduced in Table 1 and the participation rates making the heterogeneous tontine individually fair. Furthermore, we assume $\rho_i = r$.

4 Conclusion

This paper studies financial fairness in tontines with heterogeneous cohorts. The tontine design we consider is the one introduced in Milevsky and Salisbury (2016). We distinguish between collective and individual fairness, show how the equitable design in Milevsky and Salisbury (2016) can be changed to achieve collective fairness and demonstrate that collective fairness widely implies individual fairness. In the second part of this paper, we consider a utilitarian social planner setting up a tontine on behalf of the heterogeneous policyholders. We solve the collective optimization problem and assume that the social planner ensures individual fairness for all individuals. Comparing the collectively optimal tontine to separate, cohort-optimized tontines as well as the proportional and natural tontine presented in Milevsky and Salisbury (2016), we find that approximately identical certainty equivalents are achieved under homogeneous log-utility preferences. However, under higher and heterogeneous degrees of relative risk aversion, the social planner and the homogeneous-cohort tontines outperform the proportional and natural tontine. In contrast to Milevsky and Salisbury (2016), we are unable to show that the benefits of pooling heterogeneous cohorts exceed the benefits of individually utility-maximizing withdrawal rates in single-cohort tontines.

As a possible extension to this article, we might assume that policyholders are not only heterogeneous in their ages and wealth levels, but, for example, also in their health status. We might for example assume that each cohort can be split into two groups, where one group has higher mortality rates (i.e. a lower health status) than the other group. We might then also assume that the health status is correlated with the wealth, i.e. less healthy individuals are also less wealthy. Such an additional degree of heterogeneity would increase the diversity in the pool and might change the relation between the social planner and the single-cohort tontines.

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A Proofs

A.1 Proof of Proposition 2.5

Note that individual fairness is equivalent to

$$\frac{P_0^j}{w_j} = 1 \text{ for all } j = 1 \ldots, L.$$

The proof now follows similar steps as the proof of Theorem 4(a) in Milevsky and Salisbury (2016). Assume that π and $\tilde{\pi}$ both fulfill individual fairness, are unequal and are not multiples of each other. We use the notation $P_0^j(\pi)$ throughout this proof to emphasize the dependence of P_0^j on the different sets of participation rates. Define $\pi(s) := s\pi + (1-s)\tilde{\pi}$. Then, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}P_0^j(\pi(s)) &= \int_0^\infty e^{-rt}\mathbb{E}\left[wd(t)\frac{\mathrm{d}}{\mathrm{d}s}\frac{\pi_j(s)}{\sum_{i=1}^L \pi_i(s)w_iN_i(t)}\mathbb{1}_{\{T_j>t\}}\right]\mathrm{d}t \\ &= \int_0^\infty e^{-rt}\mathbb{E}\left[wd(t)\frac{\sum_{i=1}^L w_iN_i(t)\left(\pi_i(s)\pi_j'(s)-\pi_j(s)\pi_i'(s)\right)}{\left(\sum_{i=1}^L \pi_i(s)w_iN_i(t)\right)^2}\mathbb{1}_{\{T_j>t\}}\right]\mathrm{d}t.\end{aligned}$$

Here, it holds

$$\begin{aligned}\pi_i(s)\pi_j'(s)-\pi_j(s)\pi_i'(s) &= \left(\tilde{\pi}_i + s\left(\pi_i-\tilde{\pi}_i\right)\right)\left(\pi_j-\tilde{\pi}_j\right) - \left(\pi_i-\tilde{\pi}_i\right)\left(\tilde{\pi}_j + s\left(\pi_j-\tilde{\pi}_j\right)\right) \\ &= \tilde{\pi}_i(\pi_j-\tilde{\pi}_j) - (\pi_i-\tilde{\pi}_i)\tilde{\pi}_j = \tilde{\pi}_i\pi_j - \pi_i\tilde{\pi}_j = \pi_j\pi_i\left(\frac{\tilde{\pi}_i}{\pi_i}-\frac{\tilde{\pi}_j}{\pi_j}\right).\end{aligned} \tag{12}$$

Now fix j such that $\tilde{\pi}_j/\pi_j$ is minimal. Then, it follows that (12) is ≥ 0 for all i and > 0 for at least one i (because $\tilde{\pi}$ is not a multiple of π). Hence, $\frac{\mathrm{d}}{\mathrm{d}s}P_0^j$ is positive for this j which implies $1 = P_0^j(\pi) < P_0^j(\tilde{\pi}) = 1$, clearly a contradiction. □

A.2 Proof of Theorem 3.1

Note that we can rewrite the expected lifetime utility of the collective as

$$\mathbb{E}\left[\int_0^\infty \sum_{i=1}^n e^{-\rho_i t}\beta_i \ln\left(wd(t)\cdot\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\right)\mathbb{1}_{\{T_i>t\}}\mathrm{d}t\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_0^\infty \sum_{i=1}^n e^{-\rho_i t}\beta_i \left(\ln(w) + \ln(d(t)) + \ln\left(\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\right)\right) \mathbb{1}_{\{T_i>t\}} \mathrm{d}t\right] \\
&= \int_0^\infty \sum_{i=1}^n e^{-\rho_i t}\beta_i \mathbb{E}\left[\left(\ln(w) + \ln(d(t)) + \ln\left(\frac{\pi_i w_i}{\sum_{j=1}^L \pi_j w_j N_j(t)}\right)\right) \mathbb{1}_{\{T_i>t\}}\right] \mathrm{d}t.
\end{aligned}
$$

The first-order condition is therefore:

$$
\sum_{i=1}^n e^{-\rho_i t}\beta_i \frac{1}{d(t)} \mathbb{E}\left[\mathbb{1}_{\{T_i>t\}}\right] = \lambda e^{-rt} \mathbb{E}\left[1 - \prod_{j=1}^n (1 - S_{x_j}(t))\right],
$$

which delivers

$$
d^*(t) = \frac{\sum_{i=1}^n e^{-\rho_i t}\beta_i \mathbb{E}\left[\mathbb{1}_{\{T_i>t\}}\right]}{\lambda e^{-rt}\mathbb{E}\left[1 - \prod_{j=1}^n (1 - S_{x_j}(t))\right]}.
$$

The Lagrangian multiplier λ is obtained from the budget constraint as follows:

$$
\begin{aligned}
1 &= \int_0^\infty e^{-rt} \frac{\sum_{i=1}^n e^{-\rho_i t}\beta_i \mathbb{E}\left[\mathbb{1}_{\{T_i>t\}}\right]}{\lambda e^{-rt}\mathbb{E}\left[1 - \prod_{j=1}^n (1 - S_{x_j}(t))\right]} \mathbb{E}\left[1 - \prod_{j=1}^n (1 - S_{x_j}(t))\right] \mathrm{d}t \\
&= \int_0^\infty \frac{\sum_{i=1}^n e^{-\rho_i t}\beta_i \mathbb{E}\left[\mathbb{1}_{\{T_i>t\}}\right]}{\lambda} \mathrm{d}t \\
\Leftrightarrow \quad \lambda &= \int_0^\infty \sum_{i=1}^n e^{-\rho_i t}\beta_i \mathbb{E}\left[\mathbb{1}_{\{T_i>t\}}\right] \mathrm{d}t.
\end{aligned}
$$

□

B Pseudo code for determining the fair participation rates

For a given withdrawal rate, the following pseudo code finds the fair participation rates $\pi_1, \dots, \pi_L$.

1. Fix a tolerance level *tol* and initialize all the necessary parameters.
2. Specify upper and lower bounds $\pi_2^u, \dots, \pi_L^u$ and $\pi_2^l, \dots, \pi_L^l$ for $\pi_2, \dots, \pi_L$ and set $\pi_i = \frac{1}{2}(\pi_i^u + \pi_i^l)$ for $i = 2, \dots, L$ such that (8) and (9) are fulfilled.

3. Compute P_0^1.

4. While $|P_0^1 - w_1| > tol$

   (a) Specify upper and lower bounds $\pi_3^u, \ldots, \pi_L^u$ and $\pi_3^l, \ldots, \pi_L^l$ for $\pi_3, \ldots, \pi_L$ and set $\pi_i = \frac{1}{2}(\pi_i^u + \pi_i^l)$ for $i = 3, \ldots, L$ such that (8) and (9) are fulfilled.

   (b) While $|P_0^2 - w_2| > tol$

      i. ... (Continue as in (a) and (b) for all $i = 3, \ldots, L-2$.)

      ii. Specify upper and lower bounds π_L^u and π_L^l for π_L and set $\pi_L = \frac{1}{2}(\pi_L^u + \pi_L^l)$ such that (8) and (9) are fulfilled.

      iii. While $|P_0^{L-1} - w_{L-1}| > tol$

         A. If $P_0^{L-1} - w_{L-1} > 0$, set $\pi_L = \pi_L^l$.

         B. If $P_0^{L-1} - w_{L-1} < 0$, set $\pi_L = \pi_L^u$.

         C. Compute P_0^{L-1}.

      iv. ... (Set all the π_i values as in B and C for the corresponding $i = L-2, \ldots, 3$.)

      v. If $P_0^2 - w_2 > 0$, set $\pi_3 = \pi_3^l$.

      vi. If $P_0^2 - w_2 < 0$, set $\pi_3 = \pi_3^u$.

      vii. Compute P_0^2.

   (c) If $P_0^1 - w_1 > 0$, set $\pi_2 = \pi_2^l$.

   (d) If $P_0^1 - w_1 < 0$, set $\pi_2 = \pi_2^u$.

   (e) Compute P_0^1.