A Joint Valuation of Premium Payment and Surrender Options in Participating Life Insurance Contracts

Hato Schmeiser, Joël Wagner

Institute of Insurance Economics
University of St. Gallen, Switzerland

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Institute of Insurance Economics
University of St.Gallen

Abstract

- Life insurance contracts typically embed, in addition to an interest rate guarantee and annual surplus participation,
  - a *paid-up option*, the right to stop payments during the contract term,
  - a *resumption option*, the right to resume payments, and,
  - a *surrender option*, the right to terminate the contract early.
- Terminal guarantees on benefits payable upon death, survival and surrender are adapted after option exercise.
- Model framework and algorithm to *jointly* value the options is developed.
- Standard principles of risk-neutral evaluation are applied; the policyholder is assumed to use an economically rational exercise strategy.
- Option value sensitivity on different contract parameters, benefit adaptation mechanisms, and exercise behavior is analyzed numerically.

Adequate risk management needed for correct valuation of options and compliance with regulation rules

- Financial crisis and variable annuity products with embedded options have highlighted the importance of the valuation and the risk management of options.
  → *E.g. Equitable Life (2000), The Hartford (2009), AXA (2009).*
- Current and planned regulation rules prescribe risk-adequate capital deposits for embedded options.
  → *E.g. Solvency II, SST, NAIC RBC, and IFRS.*
- Hypotheses on the policyholder's exercise behavior are required to evaluate option values, in particular with regard to stress-testing of products.
  → *How much more value can policyholders get by exercising differently than what has been observed?*

Competitive advantage for product design and pricing when knowing the "fair" price

- Separate inspection of embedded options with regard to pricing and risk management.
  $\rightarrow$ *Pricing flexibility through reconfiguration of the conversion mechanism of the guaranteed benefits after exercise of the option.*
- Product design and offering of products along the needs and willingness to pay of customers.
  $\rightarrow$ *Possibility of offering zero-valued options, not to be paid for by the client.*
- Adequate and "fair" pricing of embedded options since policy inception.
  $\rightarrow$ *Competitive advantage.*

Literature review of selected scientific contributions I

- Fair pricing of contracts with a guaranteed interest rate and different annual surplus participations.
  $\rightarrow$ *E.g. Bacinello (2001); Hansen and Miltersen (2002); Tanskanen and Lukkarinen (2003); Ballotta et al. (2006); Branger et al. (2010).*
- Surrender option, on top of guaranteed interest rate and surplus participation.
  $\rightarrow$ *E.g. Grosen and Jorgensen (2000); Jensen et al. (2001).*
- Surrender option in Italian life insurance contracts with single and periodic premiums, including mortality risk.
  $\rightarrow$ *E.g. Bacinello (2003a,b, 2005); Bacinello et al. (2009).*
- Surrender option in French life insurance contracts.
  $\rightarrow$ *E.g. Albizzati and Geman (1994).*
- Surrender and paid-up options with underlying stochastic interest rates and deterministic surplus participation.
  $\rightarrow$ *E.g. Herr and Kreer (1999).*

Literature review of selected scientific contributions II

- General framework analysis of surrender and paid-up options. $\rightarrow$ *E.g. Steffensen (2002).*
- Actuarial approach comparison. $\rightarrow$ *E.g. Linnemann (2003, 2004).*
- Numerical analysis of risk-neutral valuation of embedded options and broad overview on numerical techniques. $\rightarrow$ *E.g. Bauer et al. (2006); Bauer et al. (2008).*
- Analysis of paid-up options in German government-subsidized pension products based on different assumptions about the policyholder's exercise behavior. $\rightarrow$ *E.g. Kling et al. (2006).*
- Option valuation through an optimal feasible exercise strategy in the case of one Bermudan-style embedded option. $\rightarrow$ *E.g. Barraquand and Martineau (1995); Andersen (1999); Douady (2002); Kling et al. (2006).*

What is the contribution of this work?

- Model framework and methodology closest to the following:
  - Gatzert and Schmeiser (2008): paid-up and resumption options with focus on the maximal risk.
  - Bacinello (2003b): surrender option pricing.
  - Andersen (1999): option valuation algorithm for one option.
- Key features:
  - Several options are valued jointly.
    $\rightarrow$ *Most works consider options separately.*
  - Pricing through assumptions on policyholder's behavior is given.
    $\rightarrow$ *Not only assessment of maximal risk potential.*
  - Exercise through economically rational optimal strategy.
    $\rightarrow$ *Feasible strategy.*
  - Analysis of option values for various benefit conversion mechanisms in a policy assets perspective.
    $\rightarrow$ *Perspective allows for asset transparency in comparison with reserve-linked modeling.*

Structure of this talk

- **Introduction of the model framework**
  - Basic contract $\mathcal{B}$ including interest guarantee and surplus participation.
  - Contract $\mathcal{P}$ with a paid-up option $\mathcal{O}^{\mathcal{P}}$.
  - Contract $\mathcal{R}$ with a combined paid-up and resumption option $\mathcal{O}^{\mathcal{R}}$.
  - Contract $\mathcal{S}$ featuring a surrender option $\mathcal{O}^{\mathcal{S}}$.
  - Contract $\mathcal{Q}$ with a combined paid-up and surrender option $\mathcal{O}^{\mathcal{Q}}$.
- **Option valuation techniques and exercise behavior hypotheses**
  - Maximum over all times of exercise of the option payoff.
  - *Option value* reached using an optimal admissible exercise strategy.
  - Upper bound of the option payoff for any exercise strategy (risk).
- **Numerical results and discussion**
  - Option values for different contracts and parameters.
  - Sensitivity analysis of the model.
  - Comparison with Gatzert and Schmeiser (2008) and Bacinello (2003b).

Basic contract $\mathcal{B}$

- Life insurance contract with periodic premium payments featuring
  - an interest rate guarantee g, and,
  - an annual surplus participation, i.e. fraction α of the surplus.
- **Premium payments** B_{t-1}, $t = 1, \ldots, T$, are paid annually at the beginning of the tth policy year given the insured remains alive.
  $\rightarrow$ *Annual premium payments $B_t \equiv B$ are supposed constant.*
- **Mortality statistics:** financial risk and mortality risk are uncorrelated; mortality risk is eliminated by writing a sufficiently large number of contracts (actuarial practice).
  $\rightarrow$ *Mortality risk does not contain systematic risk.*
- **Survival and death probabilities:**
  - ${}_tp_x$: x-year-old policyholder survives for the next t years.
  - ${}_tq_x = 1 - {}_tp_x$: x-year-old dies within the next t years.
  - q_{x+t} (resp. p_{x+t}): $(x+t)$-year-old policyholder will die within the next year, between t and $t+1$ (resp. survive one more year, the tth year).
- Guarantees are on benefits payable upon *death* and *survival*.
  $\rightarrow$ *Contract $\mathcal{B}$ non-surrenderable and without paid-up option.*

Death benefit

- In case of death during the tth year of the contract, the assigned policyholder's beneficiary receives $\Upsilon_t^{\mathcal{B}} \equiv \Upsilon^{\mathcal{B}}$ at the end of the year.
- Actuarial equivalence principle: the expected value of the payments to the insured equals the expected premium payments from the insured:

$$B \sum_{t=0}^{T-1} {}_t p_x \, (1+g)^{-t} = \Upsilon^{\mathcal{B}} \left(\sum_{t=0}^{T-1} {}_t p_x \, q_{x+t} \, (1+g)^{-(t+1)} + {}_T p_x \, (1+g)^{-T} \right).$$

- The death benefit is given by

$$\Upsilon^{\mathcal{B}} = \frac{B \sum_{t=0}^{T-1} {}_t p_x \, (1+g)^{-t}}{\sum_{t=0}^{T-1} {}_t p_x \, q_{x+t} \, (1+g)^{-(t+1)} + {}_T p_x \, (1+g)^{-T}}.$$

Survival benefit I

- In case of survival until maturity T, the insurer pays out the accumulated policy assets $A_T^{\mathcal{B}}$.
  $\rightarrow$ *Including guaranteed interest and surplus participation.*
- Annual premium payments B are split into
  - Premium B_{t-1}^R for term life insurance, used to buy insurance to cover the difference between the guaranteed death benefit $\Upsilon^{\mathcal{B}}$ and the available policy assets $A_{t-1}^{\mathcal{B}}$. For the tth year, fair pricing implies that

$$B_{t-1}^R = q_{x+t-1} \max(\Upsilon^{\mathcal{B}} - A_{t-1}^{\mathcal{B}}, 0).$$

  - Savings premium $B_{t-1}^A = B - B_{t-1}^R$.
    $\rightarrow$ *Credited to the policy assets at the beginning of the tth year.*
- $A_{t-1}^{\mathcal{B}}$ and B_{t-1}^A annually earn the greater of the the guaranteed interest rate g or a fraction α of the annual surplus $(S_t/S_{t-1} - 1)$.

Survival benefit II

- Life insurer's investment portfolio S_t follows a geometric Brownian motion given a complete, perfect, and frictionless market.
  For $t = 1, \ldots, T$, S_t is defined by

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t^P,$$

  with deterministic asset drift μ, volatility σ, and a P-Brownian motion W^P.

- Under the risk-neutral unique equivalent martingale measure Q, the drift of the process is the risk-free interest rate r, and

$$S_t = S_{t-1} \exp\left[r - \sigma^2/2 + \sigma(W_t^Q - W_{t-1}^Q)\right],$$

  with Q-Brownian motion W^Q and initial condition S_0 (see, e.g., Björk (2004)).

Survival benefit III

- Finally, the accumulated policy assets $A_t^{\mathcal{B}}$, $t = 1, \dots, T$, can be calculated as follows,

$$\begin{aligned} A_t^{\mathcal{B}} &= \left(A_{t-1}^{\mathcal{B}} + {}_{t-1}p_x \, B_{t-1}^{A}\right)(1 + \max\left[g, \alpha(S_t/S_{t-1} - 1)\right]) \\ &= \left(A_{t-1}^{\mathcal{B}} + {}_{t-1}p_x \left[B - q_{x+t-1} \max(\Upsilon^{\mathcal{B}} - A_{t-1}^{\mathcal{B}}, 0)\right]\right) \\ &\qquad \cdot (1 + \max\left[g, \alpha(S_t/S_{t-1} - 1)\right]), \end{aligned}$$

with $A_0^{\mathcal{B}} = 0$.

Contract payoff I

- Payoff $P_T^{\mathcal{B}}$ at maturity T is determined by the payments to the insured less the premium payments, compounded with the risk-free rate r:

$$P_T^{\mathcal{B}} = \sum_{t=0}^{T-1} \Upsilon^{\mathcal{B}} \, {}_t p_x \, q_{x+t} \, e^{r(T-t-1)} + {}_T p_x \, A_T^{\mathcal{B}} - \sum_{t=0}^{T-1} B \, {}_t p_x \, e^{r(T-t)}.$$

- Let $\Pi_0^{\mathcal{B}}$ denote the net present value of the contract payoff $P_T^{\mathcal{B}}$ at time $t = 0$. With E_t^Q denoting the conditional expected value with respect to the probability measure Q under the information available in t, one gets

$$\Pi_0^{\mathcal{B}} = E_0^Q \left(e^{-rT} P_T^{\mathcal{B}} \right).$$

Contract payoff II

- Contract is *fair* if the present value of the benefits under the risk neutral martingale measure is equal to the present value of premiums paid by the policyholder (see, e.g., Doherty and Garven (1986)):

$$\Pi_0^{\mathcal{B}} = 0.$$

- All other parameters given, the annual surplus participation parameter α can be calibrated to obtain fair contracts.
  $\rightarrow$ *I.e. the two options in the basic contract, interest guarantee and surplus participation, are covered by the annual premium payments.*

Contract $\mathcal{P}$ with paid-up option and death benefit

- Contract $\mathcal{P}$ based on basic contract $\mathcal{B}$.
- Paid-up option $\mathcal{O}^{\mathcal{P}}$: policyholder has the right to exercise $\mathcal{O}^{\mathcal{P}}$ (once) annually until maturity. $\rightarrow$ *$\mathcal{O}^{\mathcal{P}}$ is a Bermudan-style option.*
- After exercising $\mathcal{O}^{\mathcal{P}}$ at time $t = \tau$, $\tau = 1, \dots, T-1$, denoted by $\mathcal{O}^{\mathcal{P}(\tau)}$, and given the policyholder is still alive, the terminal benefits are adjusted according to a mechanism described below.
- **Adjusted death benefit** calculated by taking the accumulated policy assets $A_\tau^{\mathcal{B}}$ as single premium for a new contract,

$$\Upsilon^{\mathcal{P}(\tau)} = \frac{A_\tau^{\mathcal{B}}(1+\gamma_\tau^{\mathcal{P}})}{\sum_{t=\tau}^{T-1} {}_{t-\tau}p_{x+\tau}\, q_{x+t}\, (1+g)^{-(t-\tau+1)} + {}_{T-\tau}p_{x+\tau}\, (1+g)^{-(T-\tau)}},$$

with $\gamma_\tau^{\mathcal{P}}$ a model parameter, with default value 0.
$\rightarrow$ *Parameter represents flexibility to adapt benefits.*

Survival benefit and contract payoff

- **Adjusted survival benefit** given by $A_T^{\mathcal{P}(\tau)}$, where $A_t^{\mathcal{P}(\tau)}$, $t = \tau, \ldots, T$, denote the accumulated policy assets over time after exercising $\mathcal{O}^{\mathcal{P}}$ at time τ:

$$\begin{aligned} A_t^{\mathcal{P}(\tau)} &= \left(A_{t-1}^{\mathcal{P}(\tau)} - {}_{t-1}p_x \, q_{x+t-1} \max(\Upsilon^{\mathcal{P}(\tau)} - A_{t-1}^{\mathcal{P}(\tau)}, 0) \right) \\ &\quad \cdot (1 + \max\left[g, \alpha(S_t/S_{t-1} - 1)\right]), \end{aligned}$$

  with $A_\tau^{\mathcal{P}(\tau)} = A_\tau^{\mathcal{B}}(1 + \gamma_\tau^{\mathcal{P}})$.

- **Contract payoff** at maturity T, in the case the paid-up option is exercised in τ, is given by

$$\begin{aligned} P_T^{\mathcal{P}(\tau)} &= \sum_{t=0}^{\tau-1} \Upsilon^{\mathcal{B}} \, {}_tp_x \, q_{x+t} \, e^{r(T-t-1)} + \sum_{t=\tau}^{T-1} \Upsilon^{\mathcal{P}(\tau)} \, {}_tp_x \, q_{x+t} \, e^{r(T-t-1)} \\ &\quad + {}_Tp_x \, A_T^{\mathcal{P}(\tau)} - \sum_{t=0}^{\tau-1} B \, {}_tp_x \, e^{r(T-t)}. \end{aligned}$$

Paid-up option value

- The value at time $t = 0$ of the paid-up option $\mathcal{O}^{\mathcal{P}(\tau)}$ exercised at time τ can be residually determined by the difference of the contract values of $\mathcal{P}$ and $\mathcal{B}$:

$$\Pi_0(\mathcal{O}^{\mathcal{P}(\tau)}) = E_0^Q \left[e^{-rT} \left(P_T^{\mathcal{P}(\tau)} - P_T^{\mathcal{B}} \right) \right] = E_0^Q \left[e^{-r\tau} \, {}_{\tau}p_x \, C_\tau(\mathcal{O}^{\mathcal{P}(\tau)}) \right],$$

where

$$\begin{aligned} C_\tau(\mathcal{O}^{\mathcal{P}(\tau)}) \quad &= \quad \sum_{t=\tau}^{T-1} \left(\Upsilon^{\mathcal{P}(\tau)} - \Upsilon^{\mathcal{B}} \right) {}_{t-\tau}p_{x+\tau} \, q_{x+t} \, e^{-r(t-\tau+1)} \\ &\quad + \left(A_T^{\mathcal{P}(\tau)} - A_T^{\mathcal{B}} \right) {}_{T-\tau}p_{x+\tau} \, e^{-r(T-\tau)} + \sum_{t=\tau}^{T-1} B \, {}_{t-\tau}p_{x+\tau} \, e^{-r(t-\tau)}. \end{aligned}$$

- Since the basic contract is fair, we have $\Pi_0^{\mathcal{P}(\tau)} = \Pi_0(\mathcal{O}^{\mathcal{P}(\tau)})$.

Contract $\mathcal{R}$ with paid-up and resumption options I

- Contract $\mathcal{R}$ based on contract $\mathcal{P}$ with paid-up option.
- Combined paid-up and resumption option $\mathcal{O}^{\mathcal{R}}$: right to stop premium payments at time $\tau = 1, \ldots, T-1$, and to resume payments at time $\nu = \tau + 1, \ldots, T-1$.
  $\rightarrow$ $\mathcal{O}^{\mathcal{R}(\tau,\nu)}$ *denotes the combined exercise at times* τ *and* ν*.*
- **Death benefit**

$$\Upsilon^{\mathcal{R}(\tau,\nu)} = \frac{A_{\nu}^{\mathcal{P}(\tau)}(1+\gamma_{\nu}^{\mathcal{R}}) + B \sum_{t=\nu}^{T-1} {}_{t-\nu}p_{x+\nu} \, (1+g)^{-(t-\nu)}}{\sum_{t=\nu}^{T-1} {}_{t-\nu}p_{x+\nu} \, q_{x+t} \, (1+g)^{-(t-\nu+1)} + {}_{T-\nu}p_{x+\nu} \, (1+g)^{-(T-\nu)}},$$

with $\gamma_{\nu}^{\mathcal{R}}$ a model parameter, with default value 0.

Contract $\mathcal{R}$ with paid-up and resumption options II

- **Survival benefit** given by $A_T^{\mathcal{R}(\tau,\nu)}$, with $A_t^{\mathcal{R}(\tau,\nu)}$, $t = \nu + 1, \ldots, T$:

$$\begin{aligned} A_t^{\mathcal{R}(\tau,\nu)} &= \left(A_{t-1}^{\mathcal{R}(\tau,\nu)} + {}_{t-1}p_x \left[B - q_{x+t-1} \max(\Upsilon^{\mathcal{R}(\tau,\nu)} - A_{t-1}^{\mathcal{R}(\tau,\nu)}, 0) \right] \right) \\ &\quad \cdot \left(1 + \max\left[g, \alpha(S_t/S_{t-1} - 1)\right]\right), \end{aligned}$$

with $A_\nu^{\mathcal{R}(\tau,\nu)} = A_\nu^{\mathcal{P}(\tau)}(1 + \gamma_\nu^{\mathcal{R}})$.

- **Contract payoff** at maturity T:

$$\begin{aligned} P_T^{\mathcal{R}(\tau,\nu)} &= \sum_{t=0}^{\tau-1} \Upsilon^{\mathcal{B}} \, {}_t p_x \, q_{x+t} \, e^{r(T-t-1)} + \sum_{t=\tau}^{\nu-1} \Upsilon^{\mathcal{P}(\tau)} \, {}_t p_x \, q_{x+t} \, e^{r(T-t-1)} \\ &\quad + \sum_{t=\nu}^{T-1} \Upsilon^{\mathcal{R}(\tau,\nu)} \, {}_t p_x \, q_{x+t} \, e^{r(T-t-1)} + {}_T p_x \, A_T^{\mathcal{R}(\tau,\nu)} \\ &\quad - \sum_{t=0}^{\tau-1} B \, {}_t p_x \, e^{r(T-t)} - \sum_{t=\nu}^{T-1} B \, {}_t p_x \, e^{r(T-t)}. \end{aligned}$$

Contract $\mathcal{R}$ with paid-up and resumption options III

- **Option value** at $t = 0$:

$$\Pi_0(\mathcal{O}^{\mathcal{R}(\tau,\nu)}) = E_0^Q\left[e^{-rT}\left(P_T^{\mathcal{R}(\tau,\nu)} - P_T^{\mathcal{B}}\right)\right] = E_0^Q\left[e^{-r\tau}\ {}_{\tau}p_x\, C_\tau(\mathcal{O}^{\mathcal{R}(\tau,\nu)})\right],$$

where

$$\begin{aligned} C_\tau(\mathcal{O}^{\mathcal{R}(\tau,\nu)}) &= \sum_{t=\tau}^{\nu-1}\left(\Upsilon^{\mathcal{P}(\tau)} - \Upsilon^{\mathcal{B}}\right)\ {}_{t-\tau}p_{x+\tau}\, q_{x+t}\, e^{-r(t-\tau+1)} \\ &\quad + \sum_{t=\nu}^{T-1}\left(\Upsilon^{\mathcal{R}(\tau,\nu)} - \Upsilon^{\mathcal{B}}\right)\ {}_{t-\tau}p_{x+\tau}\, q_{x+t}\, e^{-r(t-\tau+1)} \\ &\quad + \left(A_T^{\mathcal{R}(\tau,\nu)} - A_T^{\mathcal{B}}\right)\ {}_{T-\tau}p_{x+\tau}\, e^{-r(T-\tau)} \\ &\quad + \sum_{t=\tau}^{\nu-1} B\ {}_{t-\tau}p_{x+\tau}\, e^{-r(t-\tau)}. \end{aligned}$$

Contract $\mathcal{S}$ with surrender option I

- Contract $\mathcal{S}$ based on basic contract $\mathcal{B}$.
- Surrender option $\mathcal{O}^{\mathcal{S}}$ exercisable at time $\theta = 1, \ldots, T-1$.
  $\rightarrow$ *Contract is terminated: death benefit annihilated, surrender benefit paid out at $t = \theta$.*
- **Surrender benefit**
  $$A_\theta^{\mathcal{S}(\theta)} = A_\theta^{\mathcal{B}}(1 + \gamma_\theta^{\mathcal{S}}),$$
  where $\gamma_\theta^{\mathcal{S}}$ is a parameter with default value 0.
- **Contract payoff** at the time of surrender $t = \theta$:

$$P_\theta^{\mathcal{S}(\theta)} = \sum_{t=0}^{\theta-1} \Upsilon^{\mathcal{B}}\, {}_t p_x\, q_{x+t}\, e^{r(\theta-t-1)} + {}_\theta p_x\, A_\theta^{\mathcal{S}(\theta)} - \sum_{t=0}^{\theta-1} B\, {}_t p_x\, e^{r(\theta-t)}.$$

Contract $\mathcal{S}$ with surrender option II

- **Option value** at $t = 0$:

$$\Pi_0(\mathcal{O}^{\mathcal{S}(\theta)}) = E_0^Q\left[e^{-r\theta}\left(P_\theta^{\mathcal{S}(\theta)} - e^{-r(T-\theta)}P_T^{\mathcal{B}}\right)\right] = E_0^Q\left[e^{-r\theta}\,{}_\theta p_x\, C_\theta(\mathcal{O}^{\mathcal{S}(\theta)})\right],$$

where

$$\begin{aligned} C_\theta(\mathcal{O}^{\mathcal{S}(\theta)}) &= -\sum_{t=\theta}^{T-1} \Upsilon^{\mathcal{B}}\,{}_{t-\theta}p_{x+\theta}\,q_{x+t}\,e^{-r(t-\tau+1)} \\ &\quad + \left(A_\theta^{\mathcal{S}(\theta)} - A_T^{\mathcal{B}}\,{}_{T-\theta}p_{x+\theta}\,e^{-r(T-\theta)}\right) \\ &\quad + \sum_{t=\theta}^{T-1} B\,{}_{t-\theta}p_{x+\theta}\,e^{-r(t-\theta)}. \end{aligned}$$

Contract $\mathcal{Q}$ with paid-up and surrender options I

- Contract $\mathcal{Q}$ based on contract $\mathcal{P}$ with paid-up option.
- Combined paid-up and surrender option $\mathcal{O}^{\mathcal{Q}}$: right to stop premium payments at time $\tau = 1, \ldots, T-1$, and to surrender the contract at time $\theta = \tau + 1, \ldots, T-1$.
- **Surrender benefit** at time θ when contract has been paid-up at $t = \tau$:
$$A_\theta^{\mathcal{Q}(\tau,\theta)} = A_\theta^{\mathcal{P}(\tau)}(1 + \gamma_\theta^{\mathcal{Q}}),$$
where $\gamma_\theta^{\mathcal{Q}}$ is a parameter with default value 0.
- **Contract payoff** at the time of surrender $t = \theta$:

$$\begin{aligned} P_\theta^{\mathcal{Q}(\tau,\theta)} &= \sum_{t=0}^{\tau-1} \Upsilon^{\mathcal{B}}\, {}_t p_x\, q_{x+t}\, e^{r(\theta-t-1)} + \sum_{t=\tau}^{\theta-1} \Upsilon^{\mathcal{P}(\tau)}\, {}_t p_x\, q_{x+t}\, e^{r(\theta-t-1)} \\ &\quad + {}_\theta p_x\, A_\theta^{\mathcal{Q}(\tau,\theta)} - \sum_{t=0}^{\tau-1} B\, {}_t p_x\, e^{r(\theta-t)}. \end{aligned}$$

Contract $\mathcal{Q}$ with paid-up and surrender options II

- **Option value** at $t = 0$:

$$\Pi_0(\mathcal{O}^{\mathcal{Q}(\tau,\theta)}) = E_0^Q\left[e^{-r\theta}\left(P_\theta^{\mathcal{Q}(\tau,\theta)} - e^{-r(T-\theta)}P_T^{\mathcal{B}}\right)\right] = E_0^Q\left[e^{-r\theta}\ {}_\theta p_x\ C_\theta(\mathcal{O}^{\mathcal{Q}(\tau,\theta)})\right],$$

where

$$\begin{aligned} C_\theta(\mathcal{O}^{\mathcal{Q}(\tau,\theta)}) &= \sum_{t=\tau}^{\theta-1}\left(\Upsilon^{\mathcal{P}(\tau)} - \Upsilon^{\mathcal{B}}\right)\ {}_{t-\theta}p_{x+\theta}\ q_{x+t}\ e^{-r(t-\tau+1)} \\ &\quad - \sum_{t=\theta}^{T-1}\Upsilon^{\mathcal{B}}\ {}_{t-\theta}p_{x+\theta}\ q_{x+t}\ e^{-r(t-\tau+1)} \\ &\quad + \left(A_\theta^{\mathcal{Q}(\tau,\theta)} - A_T^{\mathcal{B}}\ {}_{T-\theta}p_{x+\theta}\ e^{-r(T-\theta)}\right) \\ &\quad + \sum_{t=\tau}^{T-1} B\ {}_{t-\theta}p_{x+\theta}\ e^{-r(t-\theta)}. \end{aligned}$$

Option valuation I

- Valuation of options is connected to the policyholder's exercise behavior.
- Different ways to approach the valuation of the embedded options, e.g.,
  - the maximum over all possible times of exercise of the expected value of the option payoff,
  - the *option value* that can be reached using an optimal admissible exercise strategy,
  - the upper bound of the option payoff for any exercise strategy.
- 1st and 3rd approaches suppose the policyholder to know the future for choosing the optimal time of exercise.
  $\rightarrow$ *This is not, in practice, a feasible strategy.*
- 3rd approach assesses the maximal risk potential on a possible path, see, e.g., Kling et al. (2006), Gatzert and Schmeiser (2008).
  $\rightarrow$ *Can be helpful for stress-testing of products.*

Maximum of expected option payoff

- Since the contract payoff is defined explicitly, the maximum over all exercise time(s) can be evaluated
- **Contract with one option**, $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}\}$:

$$\Pi_0(\mathcal{O}^{\mathcal{X}(\tau^\star)}) = \max_{\tau=1,\ldots,T-1} \left[\Pi_0(\mathcal{O}^{\mathcal{X}(\tau)})\right],$$

  where $\tau^\star$ denotes the exercise time maximizing $\Pi_0(\mathcal{O}^{\mathcal{X}(\tau)})$.
- **Contract with two options**, $\mathcal{Y} \in \{\mathcal{R}, \mathcal{Q}\}$:

$$\Pi_0(\mathcal{O}^{\mathcal{Y}(\tau^\star,\nu^\star)}) = \max_{\substack{\tau=1,\ldots,T-1 \\ \nu=\tau+1,\ldots,T-1}} \left[\Pi_0(\mathcal{O}^{\mathcal{Y}(\tau,\nu)})\right],$$

  where $\tau^\star$ and $\nu^\star$ denote the exercise times maximizing $\Pi_0(\mathcal{O}^{\mathcal{Y}(\tau,\nu)})$.

Optimal admissible exercise strategy

- Option valuation through considering a policyholder who follows an exercise strategy that maximizes the option value given the information available at the exercise date(s).
- Approach leads to an optimal stopping problem that can be solved using, for example, Monte Carlo simulation methods, as is done, e.g., in Andersen (1999), Douady (2002) or Kling et al. (2006).
- Whenever the underlying model for the economy has only one source of risk, an optimal admissible strategy for exercising Bermudan-style options can be found by looking only at the exercise value of the option, see Douady (2002).
  $\rightarrow$ *This is apparently equivalent to considering only the current value of the accumulated policy assets in the present framework.*
- Extension of Kling et al. (2006) in a setting of a contract with two options.

Case of a contract with one option I

- Contract $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}\}$ with one option $\mathcal{O}^{\mathcal{X}}$ which can be exercised at times $\tau = 1, \ldots, T-1$.
- Accumulated assets A_t at times $t = 1, \ldots, T$, before exercise of the option.
- Payoff stream $C_\tau(\mathcal{O}^{\mathcal{X}(\tau)})$ of the option at time τ.
- Exercise strategy $K = K_1 \times \ldots \times K_{T-1} \subseteq \mathbb{R}^{T-1}$.
- Following this strategy, the option is exercised at time τ if and only if the option has not been exercised before and $A_\tau \in K_\tau$, i.e.,

$$\mathbf{1}_{K_\tau}(A_\tau) = 1, \quad \text{and} \quad \mathbf{1}_{K_t}(A_t) = 0, \forall t = 1, \ldots, \tau - 1.$$

Case of a contract with one option II

- Value of the option under the assumption that the investor applies *some* given strategy K is denoted by

$$\Pi_0^K(\mathcal{O}^{\mathcal{X}}) = E_0^Q\left[e^{-r\tau}\;{}_{\tau}p_x\,C_\tau(\mathcal{O}^{\mathcal{X}(\tau)})\right],$$

  where $\tau = \inf\{t \in \{1,\ldots,T-1\} \text{ such that } A_t \in K_t\}$,
  or $\tau = T$, if $A_t \notin K_t$, $\forall t \in \{1,\ldots,T-1\}$.
  $\rightarrow$ *This last case where $\tau = T$ corresponds to a strategy where the option is never exercised, and the value of the option is zero.*

- Douady (2002) and Andersen (1999) describe the so-called value threshold method to approximate an *optimal* strategy, i.e., a strategy maximizing $\Pi_0^K(\mathcal{O}^{\mathcal{X}})$ by Monte Carlo methods.
  $\rightarrow$ *Existence of an optimal strategy $K = (0;k_1] \times \ldots \times (0;k_{T-1}]$ for some $k_t \in \mathbb{R}$, $t = 1,\ldots,T-1$ is shown.*

Case of a contract with one option III

- Method uses a backward induction algorithm to determine the optimal values $k_1, \dots, k_{T-1}$.
- Calculation of a Monte Carlo approximation follows two main parts:
  - the definition of the optimal exercise strategy K (Algorithm 1), and,
  - the calculation of the approximation of the option value (Algorithm 2).

  $\rightarrow$ *See Appendix.*
- The value $\Pi_0^K(\mathcal{O}^{\mathcal{X}})$, where K is the optimal strategy, gives an approximation of the option value $\Pi_0(\mathcal{O}^{\mathcal{X}})$.

Case of a contract with two options I

- Contract $\mathcal{Y} \in \{\mathcal{R}, \mathcal{Q}\}$ with a combined option $\mathcal{O}^{\mathcal{Y}}$ exercisable at times $\tau = 1, \ldots, T-1$ and $\nu = \tau + 1, \ldots, T-1$.
- Accumulated assets A_t at times $t = 1, \ldots, T$, before exercise of the first option, and $A_s^{(\tau)}$ the assets at times $s = \tau, \ldots, T$, where τ is the exercise date of the first option.
- Payoff stream $C_\tau(\mathcal{O}^{\mathcal{Y}(\tau,\nu)})$ of the combined option at time τ.
- Exercise strategy $K = K_1 \times \ldots \times K_{T-1} \subseteq \mathbb{R}^{T-1}$: the first option is exercised at time τ if and only if the option has not been exercised before and $A_\tau \in K_\tau$, i.e.,

$$\mathbf{1}_{K_\tau}(A_\tau) = 1, \quad \text{and} \quad \mathbf{1}_{K_t}(A_t) = 0, \forall t = 1, \ldots, \tau - 1.$$

- For each exercise time $\tau = 1, \ldots, T-1$, consider a corresponding exercise strategy $L^{(\tau)} = L_{\tau+1}^{(\tau)} \times \ldots \times L_{T-1}^{(\tau)} \subseteq \mathbb{R}^{T-\tau-1}$ with

$$\mathbf{1}_{L_\nu^{(\tau)}}(A_\nu^{(\tau)}) = 1, \quad \text{and} \quad \mathbf{1}_{L_s^{(\tau)}}(A_s^{(\tau)}) = 0, \forall s = \tau + 1, \ldots, \nu - 1.$$

Case of a contract with two options II

- The value of the option under the assumption that the investor applies some given strategy K, followed by some strategy $L^{(\tau)}$ (for τ determined by K), is denoted by

$$\Pi_0^{K,L}(\mathcal{O}^{\mathcal{Y}}) = E_0^Q\left[e^{-r\tau}\ {}_{\tau}p_x\ C_\tau(\mathcal{O}^{\mathcal{Y}(\tau,\nu)})\right],$$

where $\tau = \inf\{t \in \{1,\ldots,T-1\} \text{ such that } A_t \in K_t\}$,
or $\tau = T$, if $A_t \notin K_t$, $\forall t \in \{1,\ldots,T-1\}$, and
where $\nu = \inf\left\{s \in \{\tau+1,\ldots,T-1\} \text{ such that } A_s^{(\tau)} \in L_s^{(\tau)}\right\}$, or
$\nu = T$, if $A_s^{(\tau)} \notin L_s^{(\tau)}$, $\forall s \in \{\tau+1,\ldots,T-1\}$.

Case of a contract with two options III

- *Nested* backward induction algorithm to determine the optimal values k_t $(t = 1, \ldots, T-1)$, and $\ell_s^{(t)}$ $(t = 1, \ldots, T-1,\ s = t+1, \ldots, T)$.
- The calculation of a Monte Carlo approximation is again done in two parts,
  - the definition of the optimal exercise strategies K and $L^{(\tau)}$, $\tau = 1, \ldots, T-1$, and,
  - the calculation of the approximation of the option value.
- The value $\Pi_0^{K,L}(\mathcal{O}^{\mathcal{Y}})$ through the application of optimal exercise strategies K and $L^{(\tau)}$, gives the option value $\Pi_0(\mathcal{O}^{\mathcal{Y}})$.

Upper bound of the option payoff for any exercise behavior

- Behavioral-independent risk potential from embedded options.
  $\rightarrow$ *Policyholder knows the future and exercises the option at its maximum value.*
- Upper bound for the option price pointing out the potential hazards of offering such options, see Gatzert and Schmeiser (2008); Kling et al. (2006).
  $\rightarrow$ *This worst case from the product provider's point of view does not correspond to a realistic strategy to expect from the clients.*
- **Contract with one option**, $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}\}$:

$$\Pi_0^{\max}(\mathcal{O}^{\mathcal{X}}) = E_0^Q \left[\max_{\tau=1,\ldots,T-1} \left(e^{-r\tau} \, {}_{\tau}p_x \, \max\left(0, C_\tau(\mathcal{O}^{\mathcal{X}(\tau)})\right)\right)\right].$$

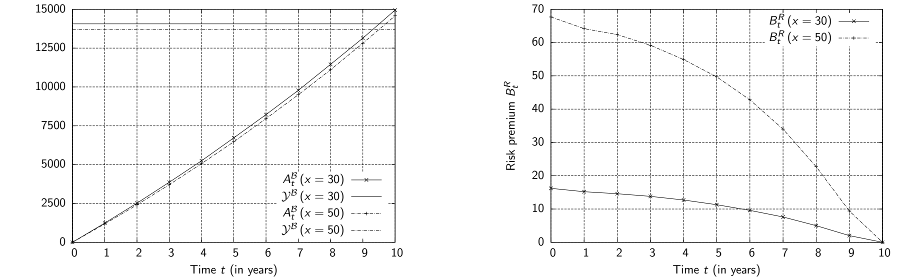
- **Contract with two options**, $\mathcal{Y} \in \{\mathcal{R}, \mathcal{Q}\}$:

$$\Pi_0^{\max}(\mathcal{O}^{\mathcal{Y}}) = E_0^Q \left[\max_{\substack{\tau=1,\ldots,T-1 \\ \nu=\tau+1,\ldots,T-1}} \left(e^{-r\tau} \, {}_{\tau}p_x \, \max\left(0, C_\tau(\mathcal{O}^{\mathcal{Y}(\tau,\nu)})\right)\right)\right].$$

Numerical simulation

- Monte Carlo simulation with antithetic variables and $N = 1\,000\,000$ different paths. $\rightarrow$ *Variance reduction by generating negatively correlated variables such that large and small outputs are counterbalanced (see, e.g., Hull (2009, p. 433)).*
- Average population mortality data derived from the Bell and Miller (2002) cohort life tables for the United States.
- Reference example with the following parametrization:
  - $x = 30$ and $x = 50$ year-old policyholders in 2010 at contract inception,
  - contract with time to maturity $T = 10$,
  - yearly premium payments $B = 1\,200$ (currency units),
  - risk-free interest rate $r = 4\%$,
  - guaranteed interest rate $g = 3\%$, and,
  - insurer's investment portfolio with volatility $\sigma = 0.20$.
- First step: calibrate the fraction α of the annual investment returns to get fair contract conditions.
  $\rightarrow$ *Standard bisection method is used to determine α for given g.*

Policy assets $A_t^{\mathcal{B}}$, death benefit $\mathcal{Y}^{\mathcal{B}}$, and risk premium B_t^R



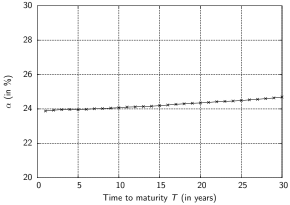
(a) Evolution of accumulated assets $A_t^{\mathcal{B}}$ and death benefit $Y^{\mathcal{B}}$ in contract $\mathcal{B}$.

(b) Evolution of risk premium B_t^R in contract $\mathcal{B}$.

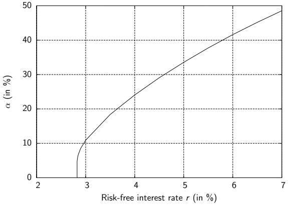
Figure: Death benefit, risk premium and accumulated assets for ages $x = 30, 50$ in basic contract $\mathcal{B}$.

$\rightarrow$ *With $x = 30$ and $x = 50$, $\alpha = 24.1\%$ and $\alpha = 29.3\%$ respectively.*
$\rightarrow$ *The standard error on the values reported for $A_t^{\mathcal{B}}$ resp. B_t^R are below 0.005% resp. 0.05%, for all $t = 0, \ldots, T$.*

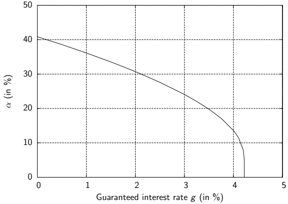
Sensitivity of fair participation coefficient α



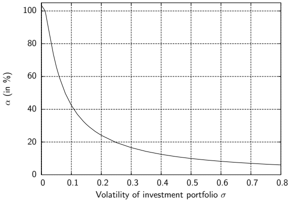
(a) α for different values T (with $r = 4\%$, $g = 3\%$, $\sigma = 0.20$).



(b) α for different values of r (with $T = 10$, $g = 3\%$, $\sigma = 0.20$).

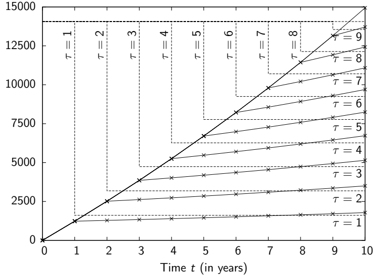


(c) α for different values of g (with $T = 10$, $r = 4\%$, $\sigma = 0.20$).

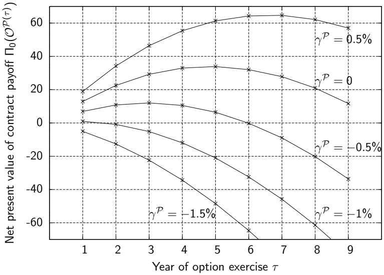


(d) α for different values of σ (with $T = 10$, $r = 4\%$, $g = 3\%$).

Policy assets $A_t^{\mathcal{P}(\tau)}$, death benefit $Y^{\mathcal{P}(\tau)}$, and $\Pi_0(\mathcal{O}^{\mathcal{P}(\tau)})$



(a) Assets $A_t^{\mathcal{B}}$, $A_t^{\mathcal{P}(\tau)}$ (solid lines), and death benefit $Y^{\mathcal{B}}$, $Y^{\mathcal{P}(\tau)}$ (dashed lines) in contract $\mathcal{P}$ for different $\tau = 1, \ldots, 9$.



(b) Sensitivity analysis of $\Pi_0(\mathcal{O}^{\mathcal{P}(\tau)})$ for different $\gamma_\tau^{\mathcal{P}} = \gamma^{\mathcal{P}} = -1.5\%, \ldots, 0.5\%$.

Figure: Analysis of contract $\mathcal{P}$ with paid-up option.

$\rightarrow$ *Parameters:* $T = 10$, $B = 1\,200$, $r = 4\%$, $\sigma = 0.20$, $g = 3\%$, $x = 30$.

$\rightarrow$ $\gamma^{\mathcal{P}} = -1\%$ *quasi annihilates the payoff for all exercise times.*

Side-by-side comparison of valuation methods

Contract duration	Item	Valuation with $\Pi_0(\mathcal{O}^{\mathcal{P}(\tau^\star)})$	Valuation with $\Pi_0^K(\mathcal{O}^{\mathcal{P}})$	Valuation with $\Pi_0^{\max}(\mathcal{O}^{\mathcal{P}})$
$T = 10$	Value	33.5 (0.2)	33.5 (0.3)	196.4 (0.2)
	Time of exercise	5	5.0 (0.0)	4.2 (0.0)
	PVP	5 535	5 538 (0.0)	4 445 (3.2)
	Value / PVP	0.6%	0.6%	4.4%
$T = 20$	Value	137.5 (0.4)	137.6 (1.1)	514.2 (0.6)
	Time of exercise	10	9.4 (0.0)	7.8 (0.0)
	PVP	10 033	9 493 (0.5)	7 247 (7.8)
	Value / PVP	1.4%	1.5%	7.1%
$T = 30$	Value	304.6 (1.0)	304.6 (2.7)	853.1 (1.5)
	Time of exercise	14	14.0 (0.0)	11.2 (0.0)
	PVP	13 010	13 015 (0.3)	9 349 (15.5)
	Value / PVP	2.3%	2.3%	9.1%

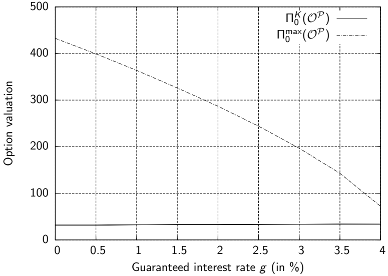
Table: Option valuation in contract $\mathcal{P}$. Reported results are for a fair underlying contract $\mathcal{B}$, $x = 30$, $r = 4\%$, $g = 3\%$ and $\sigma = 0.20$. Value in parentheses indicates standard error. PVP stands for present value of exp. premiums.

$\rightarrow$ *Maximum value of payoff reached by "optimal" exercise strategy.*

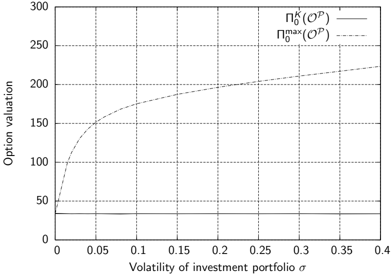
$\rightarrow$ *Option value $\Pi_0^K(\mathcal{O}^{\mathcal{P}})$ well below risk potential $\Pi_0^{\max}(\mathcal{O}^{\mathcal{P}})$.*

$\rightarrow$ *Option is worth $1-2\%$ of the expected premium payments.*

Sensitivity analysis on g and σ



(a) Valuation for different values of g (with $T = 10$, $r = 4\%$, $\sigma = 0.20$).



(b) Valuation for different values of σ (with $T = 10$, $r = 4\%$, $g = 3\%$).

Figure: Option valuation under variation of guaranteed interest rate g and volatility of investment portfolio σ.

$\rightarrow$ *Option value's independence of g and σ due to the fairness parameter α*

Detail: Sensitivity "stabilized" by α

g	α	$\Pi_0^{\mathcal{B}}$	$\Pi_0(\mathcal{O}^{\mathcal{P}(\tau^\star)})$	$\Pi_0^K(\mathcal{O}^{\mathcal{P}})$	$\Pi_0^{\max}(\mathcal{O}^{\mathcal{P}})$
1%	36.2% (fair)	0.0 (0.8)	32.4 (0.3)	32.4 (0.6)	363.6 (0.3)
3%	24.1% (fair)	0.0 (0.5)	33.5 (0.2)	33.5 (0.3)	196.4 (0.2)
1%	30.0% (fixed)	-343.4 (0.6)	285.6 (0.5)	285.6 (0.5)	463.6 (0.3)
3%	30.0% (fixed)	308.7 (0.6)	7.6 (0.1)	7.6 (0.5)	143.3 (0.1)

Table: Impact of fair and non-fair parameter α on the option valuation. For reference, values for $\Pi_0^{\mathcal{B}}$ are reported. Values reported in parentheses are the standard errors.

$\rightarrow$ *Firstly, let $g = 1\%$ and $g = 3\%$ and report valuation results for α being such that the contract $\mathcal{B}$ is fair.*
$\rightarrow$ *Secondly, fix $\alpha = 30\%$ and report the respective values.*
$\rightarrow$ *α "stabilizes" the option value.*

Option valuation for different contract parametrizations

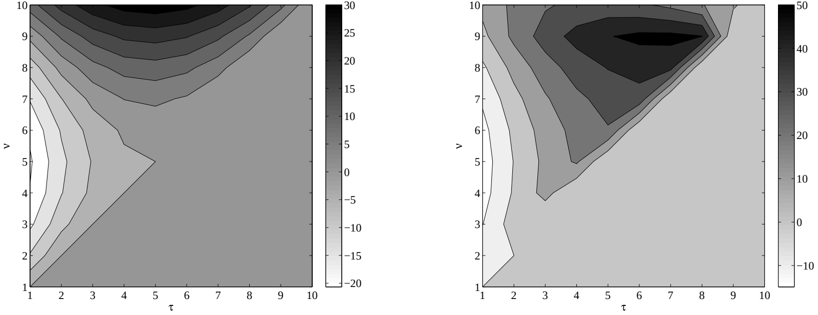
x	T	σ	g	r	α	$\Pi_0^K(\mathcal{O}^{\mathcal{P}})$	Value / PVP
30	10	0.20	3%	4%	24.1%	33.5 (0.2)	0.6%
50	10	0.20	3%	4%	29.3%	139.7 (0.4)	2.6%
30	**20**	0.20	3%	4%	42.8%	137.6 (1.1)	1.5%
30	10	**0.10**	3%	4%	42.4%	33.5 (0.3)	0.6%
30	10	0.20	**1%**	4%	36.1%	32.1 (0.6)	0.6%
30	10	0.20	**1%**	**2%**	32.4%	37.7 (0.2)	0.7%

Table: Illustration of the option value, standard error and ratio of the option value to the expected premium payments for the contract $\mathcal{P}$, with respect to different values of the age of the policyholder x, the contract duration T, the volatility of the investment portfolio σ, the guaranteed interest rate g, the risk-free interest rate r, and the determined participation rate α for the underlying contract $\mathcal{B}$ to be fair. Annual premium payments are $B = 1\,200$.

$\rightarrow$ *x and T have a high importance for the option value.*

$\rightarrow$ *Variations of σ, g, and r are counter-balanced by α.*

Option payoff $\Pi_0(\mathcal{O}^{\mathcal{R}(\tau,\nu)})$ as a function of τ and ν



(a) $\Pi_0(\mathcal{O}^{\mathcal{R}(\tau,\nu)})$ with $\gamma_\nu^{\mathcal{R}} = 0.0\%$.

(b) $\Pi_0(\mathcal{O}^{\mathcal{R}(\tau,\nu)})$ with $\gamma_\nu^{\mathcal{R}} = 0.5\%$.

Figure: $\Pi_0(\mathcal{O}^{\mathcal{R}(\tau,\nu)})$ and sensitivity with $\gamma_\nu^{\mathcal{R}}$ as a function of τ and ν.

$\rightarrow$ *Triangle $\nu \leq \tau$ is zero as resumption cannot be before paid-up.*

$\rightarrow$ *Boundaries $\tau = T$, $\nu = T$ correspond to no-exercise situations.*

$\rightarrow$ *Times and value of maximum payoff change with $\gamma_\nu^{\mathcal{R}}$.*

Option valuation in contract $\mathcal{R}$

Contract duration	Item	Valuation with $\Pi_0(\mathcal{O}^{\mathcal{R}(\tau^\star,\nu^\star)})$	Valuation with $\Pi_0^{K,L}(\mathcal{O}^{\mathcal{R}})$	Valuation with $\Pi_0^{\max}(\mathcal{O}^{\mathcal{R}})$
$T = 10$	Value	33.5 (0.2)	33.5 (0.2)	198.1 (0.2)
	Time of 1st exercise	5	5.0 (0.0)	3.9 (0.0)
	Times of 2nd exercise	10	10.0 (0.0)	9.6 (0.0)
	PVP	5 535	5 538 (0.0)	4 493 (3.1)
	Value / PVP	0.6%	0.6%	4.4%

Table: Option valuation in contract $\mathcal{R}$.

Reported results are for a fair underlying contract $\mathcal{B}$, $x = 30$, $r = 4\%$, $g = 3\%$ and $\sigma = 0.20$.

$\rightarrow$ *Maximal values when $\nu = T = 10$, i.e. second option is best never exercised ($\gamma_\nu^{\mathcal{R}} = 0.0\%$).*
$\rightarrow$ *Same values as with paid-up option only: $\Pi_0^{K,L}(\mathcal{O}^{\mathcal{R}}) = \Pi_0^{K}(\mathcal{O}^{\mathcal{P}})$, i.e. resumption option value is zero.*
$\rightarrow$ *Swiss life insurance contract: $\Pi_0^{K,L}(\mathcal{O}^{\mathcal{R}})|_{\nu=\min(\tau+2,T)}$= 20.7 (0.3) with $\tau = 8.0$, $\nu = 10.0$. PVP is 8 386 and ratio option value / PVP is 0.3%.*

Option valuation in contract $\mathcal{S}$

Contract duration	Item	Valuation with $\Pi_0(\mathcal{O}^{\mathcal{S}(\tau^\star)})$	Valuation with $\Pi_0^K(\mathcal{O}^{\mathcal{S}})$	Valuation with $\Pi_0^{\max}(\mathcal{O}^{\mathcal{S}})$
$T = 10$	Value	0.2 (0.3)	0.6 (0.4)	303.9 (0.2)
	Time of exercise	8	7.7 (0.0)	5.2 (0.0)
	PVP	8 346	8 069 (1.1)	5 493 (3.0)
	Value / PVP	0.0%	0.0%	5.5%

Table: Option valuation in contract $\mathcal{S}$.

Reported results are for a fair underlying contract $\mathcal{B}$, $x = 30$, $r = 4\%$, $g = 3\%$ and $\sigma = 0.20$.

$\rightarrow$ $\Pi_0^K(\mathcal{O}^{\mathcal{S}})$ *is zero within the considered modeling frame.*

$\rightarrow$ *However risk situation when offering the contract $\mathcal{S}$ is not alike offering a contract $\mathcal{B}$ since the non-zero upper bound $\Pi_0^{\max}(\mathcal{O}^{\mathcal{S}})$.*

Sensitivity analysis of $\Pi_0(\mathcal{O}^{\mathcal{S}(\theta)})$ for different $\gamma_\theta^{\mathcal{S}}$

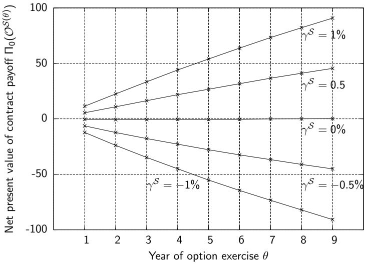
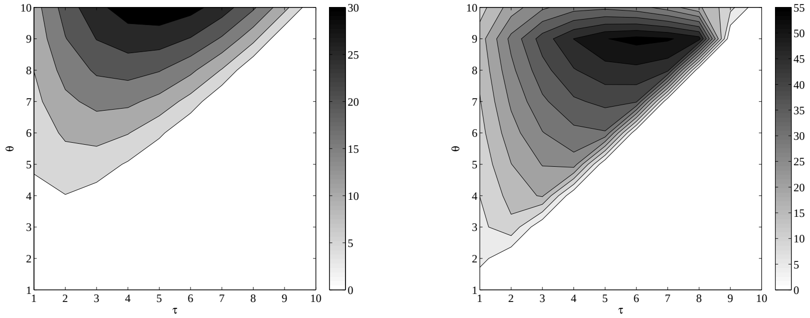


Figure: $\Pi_0(\mathcal{O}^{\mathcal{S}(\theta)})$ for different $\gamma_\theta^{\mathcal{S}} = \gamma^{\mathcal{S}} = -1.0\%, \ldots, 1.0\%$.

$\rightarrow$ *In practice, many contracts set* $\gamma_\theta^{\mathcal{S}} = -1$ *for the first two years, and zero for the following years.*
$\rightarrow$ *Comparison with examples in Bacinello (2003b): models can be calibrated through setting the parameter* $\gamma_\theta^{\mathcal{S}}$. *[Reserve-linked modeling corresponds in this case to an additional (almost negligible) loading* $< 0.1\%$ *on the assets.]*

Option payoff $\Pi_0(\mathcal{O}^{\mathcal{Q}(\tau,\theta)})$ as a function of τ and θ



(a) $\Pi_0(\mathcal{O}^{\mathcal{Q}(\tau,\theta)})$ with $\gamma_\theta^{\mathcal{Q}} = 0.0\%$.

(b) $\Pi_0(\mathcal{O}^{\mathcal{Q}(\tau,\theta)})$ with $\gamma_\theta^{\mathcal{Q}} = 0.5\%$.

Figure: $\Pi_0(\mathcal{O}^{\mathcal{Q}(\tau,\theta)})$ and sensitivity with $\gamma_\theta^{\mathcal{Q}}$ as a function of τ and θ.

$\rightarrow$ $\Pi_0(\mathcal{O}^{\mathcal{Q}(\tau,\theta)})$ *maximal for* $\tau = 5$, $\theta = 10$*; maximum not higher than in the contract* $\mathcal{P}$*.*

$\rightarrow$ $\gamma_\nu^{\mathcal{R}} = \gamma^{\mathcal{R}}$ *to* 0.5%*: payoff is maximal for* $\tau = 6$, $\theta = 9$*.*

Option valuation in contract $\mathcal{Q}$

Contract duration	Item	Valuation with $\Pi_0(\mathcal{O}^{\mathcal{Q}(\tau^\star,\theta^\star)})$	Valuation with $\Pi_0^{K,L}(\mathcal{O}^{\mathcal{Q}})$	Valuation with $\Pi_0^{max}(\mathcal{O}^{\mathcal{Q}})$
$T = 10$	Value	33.5 (0.2)	33.5 (0.2)	283.1 (0.2)
	Time of 1st exercise	5	5.0 (0.0)	4.7 (0.0)
	Times of 2nd exercise	10	10.0 (0.0)	5.7 (0.0)
	PVP	5 535	5 538 (0.0)	4 998 (2.9)
	Value / PVP	0.6%	0.6%	5.7%

Table: Option valuation in contract $\mathcal{Q}$.

Reported results are for a fair underlying contract $\mathcal{B}$, $x = 30$, $r = 4\%$, $g = 3\%$ and $\sigma = 0.20$.

$\rightarrow$ $\Pi_0^{K,L}(\mathcal{O}^{\mathcal{Q}}) = \Pi_0^K(\mathcal{O}^{\mathcal{P}})$, *hence the surrender option value is again zero.*
$\rightarrow$ $\mathcal{R}$ *and* $\mathcal{Q}$ *are very similar since the additional option offered is zero-valued (with* $\gamma_\tau^{\mathcal{R}} = \gamma_\theta^{\mathcal{Q}} = 0$*).*
$\rightarrow$ *Joint valuation is in general not equal to the sum of the single options:*

- *Contract* $\mathcal{P}$ *with* $\gamma^{\mathcal{P}} = 0.0\%$: $\Pi_0^K(\mathcal{O}^{\mathcal{P}}) = 33.5$
- *Contract* $\mathcal{S}$ *with* $\gamma^{\mathcal{S}} = 0.5\%$: $\Pi_0^K(\mathcal{O}^{\mathcal{S}}) = 45.5$
- *Contract* $\mathcal{Q}$ *with* $\gamma^{\mathcal{Q}} = 0.5\%$: $\Pi_0^{K,L}(\mathcal{O}^{\mathcal{Q}}) = 56.0$

Conclusion

- Model framework dealing with fair and joint valuation of embedded options; implementation of a robust numerical algorithm.
- Drivers of the option values pointed out:
  - Conversion mechanism of the guaranteed benefits.
    $\rightarrow$ *Throughout the studied situations, paid-up option values are of the order of* $1-3\%$ *of the expected premium payments.*
  - Exercise behavior of the policyholder.
    $\rightarrow$ *Even though an optimal exercise behavior may not be the empirically relevant case, nor the upper bound for the inherited risk be reached in practice, the practical relevance of adequate pricing is shown to be important.*
- Substantial value of embedded options shows the necessity of appropriate pricing and adequate risk management.
  $\rightarrow$ *Values of embedded options should be detailed carefully in particular with regard to the offered benefits and under various hypotheses of the client's exercise behavior.*

Further information

- **Reference for this publication**
  H. Schmeiser, J. Wagner. A Joint Valuation of Premium Payment and Surrender Options in Particpating Life Insurance Contracts. *Working Papers on Risk Management and Insurance*, 77, 2010.

- **Author contact information**
  - H. Schmeiser hato.schmeiser@unisg.ch
  - J. Wagner joel.wagner@unisg.ch

Institute of Insurance Economics
University of St. Gallen
Kirchlistrasse 2
CH–9010 St. Gallen
Phone: +41 71 243 40 43
www.ivw.unisg.ch

References I

M.O. Albizzati and H. Geman. Interest-rate Risk Management and Valuation of the Surrender Option in Life-Insurance Policies. *Journal of Risk and Insurance*, 61(4):616–637, 1994.

L. Andersen. A Simple Approach to the Pricing of Bermudan Swaption in the Multi-Factor Libor Market Model. *Journal of Computational Finance*, 3:5–32, 1999.

A.R. Bacinello. Fair Pricing of Life Insurance Participating Policies with a Minimum Interest Rate Guaranteed. *ASTIN Bulletin*, 31(2):275–297, 2001.

A.R. Bacinello. Fair Valuation of a Guaranteed Life Insurance Participating Contract Embedding a Surrender Option. *Journal of Risk and Insurance*, 70(3):461–487, 2003a.

A.R. Bacinello. Pricing Guaranteed Life Insurance Participating Policies with Annual Premiums and Surrender Option. *North American Actuarial Journal*, 7(3):1–17, 2003b.

A.R. Bacinello. Endogenous Model of Surrender Conditions in Equity-Linked Life Insurance. *Insurance Mathematics & Economics*, 37(2):270–296, 2005.

A.R. Bacinello, E. Biffis, and P. Millossovich. Pricing Life Insurance Contracts with Early Exercise Features. *Journal of Computational and Applied Mathematics*, 233(1):27–35, 2009.

L. Ballotta, S. Haberman, and N. Wang. Guarantees in With-Profit and Unitized With-Profit Life Insurance Contracts: Fair Valuation Problem in Presence of the Default Option. *Journal of Risk and Insurance*, 73(1):97–121, 2006.

J. Barraquand and D. Martineau. Numerical Valuation of High-Dimensional Multivariate Amercian Securities. *Journal of Financial and Quantitative Analysis*, 30(3):383–405, 1995.

References II

D. Bauer, R. Kiesel, A. Kling, and J. Russ. Risk-Neutral Valuation of Participating Life Insurance Contracts. *Insurance: Mathematics and Economics*, 39(2):171–183, 2006.

D. Bauer, D. Bergmann, and R. Kiesel. On the Risk-Neutral Valuation of Life Insurance Contracts with Numerical Methods in View. In *AFIR Colloquium*, Rome, 2008.

F.C. Bell and M.L. Miller. Life Tables for the United States Social Security Area 1900-2100. *Actuarial Study, Social Security Administration*, (116), 2002.

T. Björk. *Arbitrage Theory in Continuous Time*. Oxford University Press, New York, 2004.

N. Branger, A. Mahayni, and J.C. Schneider. On the Optimal Design of Insurance Contracts with Guarantees. *Insurance: Mathematics and Economics*, 2010.

N.A. Doherty and J.R. Garven. Price Regulation in Property-Liability Insurance: A Contingent-Claims Approach. *Journal of Finance*, 41(5):1031–1050, 1986.

R. Douady. *Bermudan Option Pricing with Monte-Carlo Methods*, pages 314–328. World Scientific Publishing Co., Singapore, 2002.

N. Gatzert and H. Schmeiser. Assessing the Risk Potential of Premium Payment Options in Participating Life Insurance Contracts. *Journal of Risk and Insurance*, 75(3):691–712, 2008.

A. Grosen and P.L. Jorgensen. Fair Valuation of Life Insurance Liabilities: The Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies. *Insurance: Mathematics & Economics*, 26(1):37–57, 2000.

M. Hansen and K.R. Miltersen. Minimum Rate of Return Guarantees: the Danish Case. *Scandinavian Actuarial Journal*, 2002(4):280–318, 2002.

References III

H.-O. Herr and M. Kreer. Zur Bewertung von Optionen und Garantien bei Lebensversicherungen. *Blätter der Deutschen Gesellschaft fur Versicherungsmathematik*, 24:179–193, 1999.

J.C. Hull. *Options, Futures, and other Derivatives*. Pearson Prentice Hall, 7th edition, 2009.

B. Jensen, P.L. Jorgensen, and A. Grosen. A Finite Difference Approach to the Valuation of Path Dependent Life insurance Liabilities. *Geneva Papers on Risk and Insurance Theory*, 26 (1):57–84, 2001.

A. Kling, J. Russ, and H. Schmeiser. Analysis of Embedded Options in Individual Pension Schemes in Germany. *Geneva Risk and Insurance*, 31:43–60, 2006.

P. Linnemann. An Actuarial Analysis of Participating Life Insurance. *Scandinavian Actuarial Journal*, 2003(2):153–176, 2003.

P. Linnemann. Valuation of Participating Life Insurance Liabilities. *Scandinavian Actuarial Journal*, 2004(2):81–104, 2004.

M. Steffensen. Intervention Options in Life Insurance. *Insurance Mathematics & Economics*, 31 (1):71–85, 2002.

A.J. Tanskanen and J. Lukkarinen. Fair Valuation of Path-Dependent Participating Life Insurance Contracts. *Insurance: Mathematics and Economics*, 33(3):595–609, 2003.

Algorithm 1 Definition of the optimal exercise strategy $K = (0; k_1] \times \ldots \times (0; k_{T-1}]$.

Initialize $k_\vartheta = 0$, $\vartheta = 1, \ldots, T-1$. Let $K_T = \mathbb{R}$.
Approximate k_ϑ, $\vartheta = T-1, \ldots, 1$, as follows:
Generate N Monte Carlo paths, denoted with superscript i, of S_t^i, $t = 1, \ldots, T$.
for $\underline{\text{times } \vartheta = T-1 \text{ down to } 1}$ **do** $\{$backward induction$\}$
  for $k_\vartheta = 0$ to A^K in steps of a^K **do**
    for paths $i = 1$ to N **do**
      Initialize $\tau^i = 0$.
      for $\underline{t = \vartheta \text{ to } T}$ **do**
        if $\tau^i = 0$ and $A_t^i \in K_t^\tau = (0; k_t^\tau]$ **then**
          Let $\tau^i = t$.
        end if
      end for $\{t\}$
      Calculate $e^{-r\tau^i} \; {}_{\tau^i}p_x \; C_{\tau^i}(\mathcal{O}^{\mathcal{X}(\tau^i)})$.
    end for $\{i\}$
    Calculate $\frac{1}{N}\sum_{i=1}^N e^{-r\tau^i} \; {}_{\tau^i}p_x \; C_{\tau^i}(\mathcal{O}^{\mathcal{X}(\tau^i)})$ as an estimate of $\Pi_0^K(\mathcal{O}^{\mathcal{X}})$.
  end for $\{k_\vartheta\}$
  Choose k_ϑ from all tested values such that the estimate of $\Pi_0^K(\mathcal{O}^{\mathcal{X}})$ is maximal.

end for $\{\vartheta\}$

Algorithm 2 Calculation of the Monte Carlo approximation of $\Pi_0^K(\mathcal{O}^{\mathcal{X}})$.

Use $K = K_1 \times \ldots \times K_{T-1}$ determined through Algorithm 1. Let $K_T = \mathbb{R}$.
Generate N Monte Carlo paths, denoted with superscript i, of S_t^i, $t = 1, \ldots, T$.

for paths $i = 1$ to N **do**

  Initialize $\tau^i = 0$.

  for $\underline{t = 1 \text{ to } T}$ **do**

    if $\tau^i = 0$ and $A_t^i \in K_t$ **then** {exercise of option on path i under strategy K}

      Let $\tau^i = t$.

    end if

  end for $\{t\}$

  Calculate $e^{-r\tau^i}\ {}_{\tau^i}p_x\ C_{\tau^i}(\mathcal{O}^{\mathcal{X}(\tau^i)})$.

end for $\{i\}$

Calculate $\frac{1}{N}\sum_{i=1}^{N} e^{-r\tau^i}\ {}_{\tau^i}p_x\ C_{\tau^i}(\mathcal{O}^{\mathcal{X}(\tau^i)})$ as an estimate of $\Pi_0^K(\mathcal{O}^{\mathcal{X}})$.