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MODELING PARAMETER RISK IN PREMIUM RISK IN MULTI-YEAR INTERNAL MODELS

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Modeling Parameter Risk in Premium Risk in Multi-Year Internal Models

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Abstract:

The purpose of this paper is to illustrate the importance of modeling parameter risk in premium risk, especially when data are scarce and a multi-year projection horizon is considered. Internal risk models often integrate both process and parameter risks in modeling reserve risk, whereas parameter risk is typically omitted in premium risk, the modeling of which considers only process risk. We present a variety of methods for modeling parameter risk (asymptotic normality, bootstrap, Bayesian) with different statistical properties. We then integrate these different modeling approaches in an internal risk model and compare our results with those from modeling approaches that measure only process risk in premium risk. We show that parameter risk is substantial, especially when a multi-year projection horizon is considered and when there is only limited historical data available for parameterization (as is often the case in practice). Our results also demonstrate that parameter risk substantially influences risk-based capital and strategic management decisions, such as reinsurance. Our findings emphasize that it is necessary to integrate parameter risk in risk modeling. Our findings are thus not only of interest to academics, but of high relevance to practitioners and regulators working toward appropriate risk modeling in an enterprise risk management and solvency context.

Keywords: Non-Life Insurance, Value-Based Management, Internal Risk Models, Solvency II, Parameter Risk

1. Introduction

Under Solvency II, regulators in Europe will require insurers to use either a new *standard formula* or their own *internal risk model* in determining risk-based capital standards. Since their development in the 1990s, internal risk models have become an important tool for financial modeling and scenario analysis in the non-life and reinsurance industry. Today, they serve as a major instrument in assessing risk and return of strategic alternatives and thus support managerial decision making (see, e.g., Kaufmann et al., 2001; Eling and Toplek, 2009). Recent solvency discussions at the European level extended the use of such models to external risk management and regulation.

The standard formula and internal risk models both need to adequately reflect the insurer's risk situation, which means they need to be able to measure and evaluate every relevant risk to which the insurer is exposed. With regard to underwriting risk, premium and reserve risk are typically considered to be two separate risk category types. One of the underlying sources of risk, *prediction risk*, can be divided into *parameter risk* and *process risk* (see Cairns, 2000). The Solvency II standard formula and internal risk models integrate both process and parameter risks in modeling reserve risk, whereas parameter risk is often omitted in premium risk, the modeling of which considers only process risk. To our knowledge, this is also true in regulatory jurisdictions other than the European Union, such as the United States and Australia. This is surprising given discussion in business and academia about the importance of modeling parameter risk (see, e.g., Cairns, 2000; Borowicz and Norman, 2006a).

This paper illustrates the modeling of non-life premium risk, including parameter risk, in an enterprise risk management framework. There are several approaches to modeling parameter risk and each has its own theoretical foundation. We thus present a variety of methods (asymptotic normality, bootstrap, Bayesian approaches) with different statistical properties and compare them in an application to a non-life insurance company.

We integrate parameter risk into the multi-year internal risk model presented in Diers (2011), which was extended in Diers (2012), and compare our results with modeling approaches that only measure process risk in premium risk. The results show that parameter risk is substantial, especially when a multi-year projection horizon is considered and only limited historical data are available for model parameterization (as is often the case in practice). Our findings also indicate that parameter risk can substantially influence risk-based capital and strategic management decisions, such as reinsurance. Our findings are thus not only important for academics, but also for practitioners and regulators working toward appropriate parameter risk modeling in an enterprise risk management and solvency context. To our knowledge, there are no model approaches or studies on parameter risk for projection periods of not just one, but several, accident years; however, consideration of multiple years is crucial when thinking strategically about enterprise risk management and, indeed, Solvency II requires implementation of multi-year risk capital projections in the "Own Risk and Solvency Assessment" process (ORSA; see Elderfield, 2009; CEIOPS, 2008).

The remainder of this paper is structured as follows: Section 2 contains a brief summary of statistical estimation theory, demonstrates the sources and relevance of parameter risk in non-life insurance, and introduces the multi-year model framework for internal risk models to allow for parameter risk, where we extend the multi-year model approach presented in Diers (2011) and Diers (2012). Section 3 presents three approaches to modeling parameter risk: the asymptotic normality approach, the bootstrap method, and the Bayesian approach. Section 4 presents an application of these modeling approaches in a multi-year framework to show the influence of parameter risk on an insurance company's risk situation. Section 5 concludes.

2. Modeling Parameter Risk in Premium Risk

2.1 Premium Risk in Non-Life Insurance

Premium risk refers to the risk that future premiums will not be sufficient to cover claims. Usually, premium risk is one of the major risks for non-life insurers. Modeling premium risk using internal models involves stochastic claims models (such as a collective risk model with a frequency-severity approach) assuming claim numbers and severities with parametric probability distributions. Selecting appropriate probability distributions, together with estimating their respective parameters, is usually based on the company's own historical record of claim numbers and severities.[1] Methods using statistical estimation and testing theory as well as exploratory data analysis aim at identifying a statistical law from the past and then extrapolating it into the future (see Diers, 2007).

Internal models rely on a certain prediction process that may give rise to different sources of uncertainty—model uncertainty and prediction uncertainty, which can be divided into parameter risk and process uncertainty (see Cairns, 2000).[2] Process risk describes uncertainty from the actual random process. Parameter risk results from uncertainty in estimating the parameters of a model.

Application for steering purposes as well as reinsurance modeling may require a very granular modeling approach (on the level of business divisions, business units, individual hazards) in the internal model, requiring a large number of parameters to be specified.

This paper is limited to modeling parameter risk in univariate distributions assuming independent claim severity; the reader is referred to Borowicz and Norman (2006b) for modeling parameter risk in dependency structures.

2.2. Parameter Estimation and Risk

Typically the estimation process is based on a data set $y^{(n)} := (y_1, \ldots, y_n)$ of n observations containing historical claim sizes or numbers (on accident or underwriting period basis). For the most common claims models used in non-life insurance, $y^{(n)}$ can be treated as a sample of independent realizations y_i from a random variable Y. As we are assuming parametric distribution models, we treat the general distribution class of Y as known, that is, the distribution of Y (which we refer to as process distribution) has been fully specified except for an unknown parameter (we thus

1 One exception is catastrophe (CAT) claim modeling, which often draws on external sources due to the limits in a company's own historical record. CAT claim modeling is not covered in this paper.

2 Other sources of uncertainty include missing data and risk of change.

do not consider model risk). If this class is referred to as $\Gamma = \{F_Y(\theta): \theta \in \Theta \subseteq \Re^d\}$, element $F_Y(\theta_0)$ remains to be determined in the distribution family as corresponds to the original distribution of *Y*. Since d-dimensional parameter vector θ_0 is unknown, it will need to be deduced from the finite set of available observations. A (point) estimator $\hat{\theta}^{(n)} := \hat{\theta}^{(n)}(y_1,...,y_n)$ is a mapping from the space of all possible outcomes of the sample $y^{(n)}$(in general a subset of $\Re^n$) into the parameter space Θ of feasible parameter values; hence $\hat{\theta}^{(n)}$ compresses the sample information to one single value. The value of $\hat{\theta}^{(n)}$ is dependent on the outcome of $y^{(n)}$ and therefore itself a realization of a random variable. This yields parameter risk.

The degree of parameter risk depends on the quality of the estimation method actually used to determine $\hat{\theta}^{(n)}$(for quality criteria such as unbiasedness, consistency, and efficiency, see Witting, 1985) and the size of the data set $y^{(n)}$. In most cases, parameter risk increases as the size *n* decreases, all else equal.

The maximum likelihood method is commonly used in non-life insurance due to its superior (asymptotical) statistical properties, assuming *Y* with probability density function $f_Y(\cdot;\theta)$. Due to the independence of the sample elements, the corresponding *likelihood* of the sample $y^{(n)}$ $L(\theta; y^{(n)}) := p(y^{(n)} \mid \theta)$ can be further expressed as:

$$L(\theta; y^{(n)}) = f_Y(y_1;\theta) \cdot f_Y(y_2;\theta) \cdot ... \cdot f_Y(y_n;\theta). \quad (1)$$

The unknown parameter (set) θ_0 is estimated by

$$\hat{\theta}_{\text{ML}}^{(n)} := \arg\max_{\theta} L(\theta; y^{(n)}), \quad (2)$$

which we refer to as the *maximum likelihood estimator* (MLE; for details, see, e.g., Kiefer, 1987). Consequently, parameters estimated by the maximum likelihood method yield the highest probability for the data observed compared to any other method. The MLE is asymptotically efficient given certain regularity conditions fulfilled for distribution classes as commonly used in non-life insurance.

Parameter risk is not specific to either non-life insurance or premium risk, but inherent in any type of forecast. In modeling risk types such as reserve risk, process and parameter risk are accounted for in both the standard formula of Solvency II and in internal risk models (quantification of the prediction error either analytically or with Bayesian, bootstrap, or other simulation methods). By contrast, parameter risk is typically ignored in premium risk modeling, leaving only process risk. Changes in portfolio and practical data issues limiting the validity of records for estimating distribution parameters present a major challenge to premium risk modeling in non-life insurance.

2.3. Framework for Modeling Parameter Risk in Premium Risk

We now present the model framework for premium risk in a multi-year projection period taking parameter risk into account, while extending classical modeling approaches such as those presented in Diers (2007) and Eling and Toplek (2009). Let t denote the actual year - premium risk for a single future accident year $t+i, i = 1,\dots,m,$ is usually measured from the corresponding underwriting result UW_{t+i}:

$$UW_{t+i} = P_{t+i} - E_{t+i} - S_{t+i}, \qquad (3)$$

where S_{t+i} denotes the total ultimate claims, P_{t+I} the earned premiums, and E_{t+i} the costs.

In the following we use the multi-year model framework presented in Diers (2011), which was extended in Diers (2012). In a multi-year framework, premium risk is measured from the cumulated underwriting result

$$UW_{[t+1,t+m]} := \sum_{i=1}^{m} UW_{t+i} = \sum_{i=1}^{m} P_{t+i} - \sum_{i=1}^{m} E_{t+i} - \sum_{i=1}^{m} S_{t+i}.$$

This paper focuses on modeling the cumulated ultimate claims $S_{[t+1,t+m]} := \sum_{i=1}^{m} S_{t+i}$ with both the premiums P_{t+i} and the costs E_{t+1}, each assumed as known.[3]

Regarding claims modeling for premium risk, in internal risk models it is convenient to distinguish between catastrophe (CAT), large, and attritional claims. CAT claims generally are modeled using a different approach that combines geophysical models with external sources and therefore are not covered in this paper. Large claims are those claims that exceed a given threshold. Large claims are modeled with a collective risk model approach, whereas the bulk of attritional claims (all remaining claims below the threshold) is modeled directly as a total claim amount.

Hence the annual total claim loss S_{t+i} of any future accident year $t+i, i = 1,\ldots,m$, is calculated as the sum of large and attritional claims:

$$S_{t+i} = S_{t+i}^{L} + S_{t+i}^{A}, \text{where } S_{t+i}^{L} = \sum_{k=1}^{N_{t+i}} X_{t+i,k}. \tag{4}$$

$X_{t+i,k}, k = 1,\ldots, N_{t+i}$ denotes single large claims up to the number N_{t+i} and S_{t+i}^{A} refers to the bulk of attritional claims in an accident year $t+i$.

Our multi-year model framework extends the basic claims models to allow for stochastic parameters in order to account for parameter risk, and hence obtain a full predictive distribution of the total claim amount.

Denoting process distributions with F_X, F_N, F_{S^A} and the corresponding parameter estimators with $\hat{\theta}_X, \hat{\theta}_N, \hat{\theta}_{S^A}$, we make the following model assumptions for our multi-year claims predictive model:

- Parameters $\hat{\theta}_X, \hat{\theta}_N, \hat{\theta}_{S^A}$ are stochastic and follow certain parameter distributions.[4]

[3] Multi-year projections also typically include premium cycles, inflation, and reinsurance cycles, none of which we describe here.

[4] Note that the traditional approach of ignoring parameter risk implicitly assumes perfect knowledge about the parameters and leads to a Dirac distribution having full probability mass in the estimated parameter value.

- Annual large claim numbers $N_{t+1}, \ldots, N_{t+m}$ per future accident year *t+1,…,t+m* are conditionally independent and identically $F_N(\theta)$-distributed, conditional on $\hat{\theta}_N = \theta$.
- Single large claim sizes $X_{t+1,1}, \ldots, X_{t+1,N_{t+1}}, \ldots, X_{t+m,1}, \ldots, X_{t+m,N_{t+m}}$ to occur in future accident years *t+1,…,t+m* are conditionally independent and identically $F_X(\theta)$-distributed, given $\hat{\theta}_X = \theta$.
- Large claim numbers are independent from single large claim sizes.
- Attritional claims $S^A{}_{t+1}, \ldots, S^A{}_{t+m}$ per accident year *t+1,…,t+m* are conditionally independent and identically $F_{S^A}(\theta)$-distributed, given $\hat{\theta}_{S^A} = \theta$.
- Attritional claims are independent from large claims.

The predictive distributions in our model arise are based on a combination of process and parameter distributions, and are easy to apply in the simulation environment of internal risk models using the following two-step process:

- Step 1: The parameters are randomly drawn from the associated parameter distribution.
- Step 2: Claim sizes, claim numbers, and attritional claims are generated independently from the corresponding parametric process distributions according to the distribution parameters in the specific simulation.

Figure 1 illustrates the two-step process for claim numbers.

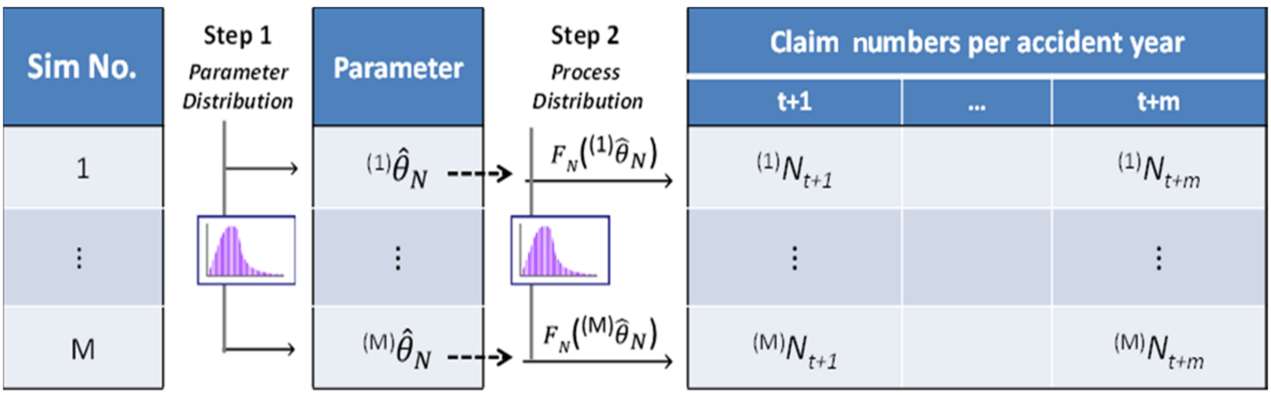

Figure 1: Two-step process for claim numbers

Note that according to our model framework, the randomly drawn parameter sets apply pathwise to each accident year in a multi-year projection; hence predicted overall claim losses from different accident years are dependent due to the common modeling of parameter risk. This is especially relevant for multi-year risk and capital projections, as it lowers the diversification benefit and balance over time.

As we assume that the process distributions F_X, F_N, F_{S^A} have already been fixed, we only have to determine the parameter distributions for $\hat{\theta}_X, \hat{\theta}_N, \hat{\theta}_{S^A}$, which is the subject of the next section.

3. Methods for Modeling Parameter Risk

Several methods for determining the parameter distributions can be found in the literature (see, e.g., Cairns, 2000; Mata, 2000). This paper will analyze three of these methods to quantify parameter risk in premium risk—the asymptotic normality approach, bootstrap methods, and the Bayesian approach (as described by Borowicz and Norman, 2006a). We briefly introduce these methods, summarize their key advantages and disadvantages, and illustrate their calculation using an example of large claim numbers modeling.

Accident year	Number of claims
2000	0
2001	1
2002	1
2003	1
2004	3
2005	2
2006	2
2007	0
2008	2
2009	3
2010	1

Table 1: Observed claim numbers

Table 1 is an example of a record of ultimate claim numbers of large claims from 11 accident years for a segment. We assume that the process distribution is given by a Poisson (λ) distribution[5] (λ>0) with density

$$f_N(k;\lambda) = \frac{\exp\{-\lambda \cdot k\}}{k!}, k = 0,1,2,...$$

The Poisson distribution is commonly used for modeling claim numbers, where the parameter λ>0 represents the expected number of claims per year. Note that in the Poisson model, mean and variance coincide.

For a given data set $y^{(n)} = (y_1, \ldots, y_n)$, the MLE for parameter λ in the Poisson model is simply the sample mean $\hat{\lambda}^{(n)} = \frac{1}{n} \cdot \sum_{i=1}^{n} y_i$. The estimator $\hat{\lambda}^{(n)}$ is unbiased, its variance is given by $\text{Var}\left[\hat{\lambda}^{(n)}\right] = \frac{\lambda}{n}$; however, for modeling purposes we need the full distribution of $\hat{\lambda}^{(n)}$. In the Poisson case, the distribution of $\hat{\lambda}^{(n)}$ does not fit a known parametric distribution class, but can be approximated and modeled with one the following methods. Note that for most distribution classes, the ML estimates in finite sample sizes have to be calculated with numerical methods; furthermore, mean and variance of MLE cannot be expressed with analytical closed-form formulas.

3.1 Asymptotic Normality Approach

This relatively simple approach to modeling parameter risk takes advantage of the fact that under the true distribution model $F_Y(\theta_0)$, commonly used estimators are asymptotically normal, which means that for the sequence of estimators $\left\{\hat{\theta}^{(n)}\right\}_{n \in N}$

$$\sqrt{n} \cdot \left(\hat{\theta}^{(n)} - \theta_0\right) \xrightarrow{d} \mathrm{N}(0, \Sigma(\theta_0)), \quad n \to \infty \text{holds,(5)}$$

[5] For simplicity, we desist from using statistical methods to select a suitable model for modeling claim frequencies here (e.g., Poisson or negative binomial).

Where $\Sigma(\theta_0)$ denotes the asymptotic variance-covariance matrix of the estimator. For the MLE, $\Sigma(\theta)$ is given by the inverse of the Fisher information matrix (see Borowicz and Norman, 2006a; for a straightforward mathematical description, see Millar, 2011).

For a finite (but sufficiently large) sample, the distribution of the MLE may be approximated by a (multivariate) normal distribution with mean $\hat{\theta}^{(n)}$ and variance-covariance matrix $n^{-1} \cdot \Sigma(\hat{\theta}^{(n)})$. In the case of a Poisson (λ) distribution, the asymptotic variance of the MLE is λ .

3.2 Bootstrap Methods

The bootstrap method, a well-known statistical re-sampling method (see Efron and Tibshirani, 1993), is another approach to quantifying parameter risk.

The basic concept lies in generating a sufficient number M of pseudo-records ${}^{(1)}y^{(n)},\ldots,{}^{(M)}y^{(n)}$ that have the same size and and are subject to the same laws of probability as the original data set $y^{(n)}$. The estimation process is then applied to each of these pseudo-data records as in estimating $\hat{\theta}^{(n)}$. This yields estimates ${}^{(1)}\hat{\theta}^{(n)},\ldots,{}^{(M)}\hat{\theta}^{(n)}$, which is referred to as the bootstrap distribution of the estimator.

It is important to distinguish between non-parametric and parametric bootstrapping methods. The non-parametric bootstrap method involves sampling with replacement from the original sample, and requires independent and identically distributed observations, whereas parametric bootstrapping involves creating new samples from $F_Y(\hat{\theta}^{(n)})$, the assumed parametric distribution class with the initially estimated parameter values.

3.3 Bayesian Approach

The Bayesian approach, as described in Borowicz and Norman (2006a), is a third alternative for modeling parameter risk. In contrast to classical statistics, the unknown distribution parameter is treated as a random variable in a Bayesian framework. This per se involves departure from the logic described in Section 2; nevertheless, the Bayesian approach can be easily incorporated into our model framework using the posterior distribution of the parameter as the parameter distribution. Consider again $y^{(n)}$ as observations from random variable *Y* and let θ be the random parameter of the distribution of *Y*. With some initial knowledge on how the unknown parameters are distributed—via specification of a prior distribution $p(\theta)$—one can produce a *posterior distribution* $p(\theta | y^{(n)})$ using Bayes's Theorem:

$$p(\theta | y^{(n)}) = \frac{1}{p(y^{(n)})} \cdot p(y^{(n)} | \theta) \cdot p(\theta). \qquad (6)$$

As $p(y^{(n)} | \theta) = L(\theta | y^{(n)})$ is the likelihood function of the sample $y^{(n)}$ according to Equation (1), the posterior distribution of the parameter is simply proportional to the likelihood function under uniform (non-informative) prior distribution $p(\theta)$ (see also Borowicz and Norman, 2006a):

$$p(\theta | y^{(n)}) \propto L(\theta | y^{(n)}). \qquad (7)$$

Considering the claim numbers example with Poisson(λ) distribution and a uniform (non-informative) prior distribution, the posterior distribution of the parameter $\theta = \lambda$ is a Gamma distribution (see Fink, 1995):

$$p(\lambda | y^{(n)}) = \frac{\lambda^{\alpha-1} exp\{-\lambda / \beta\}}{\Gamma(\alpha)\beta^{\alpha}}. (8)$$

with parameters

$$\alpha = \sum_{i=1}^{n} y_i + 1 \quad \text{and} \quad \beta = \frac{1}{n}. (9)$$

3.4 Comparison of Approaches

Although easy to implement, serious shortcomings of the asymptotic normality approach include that normal distribution can lead to non-feasible parameter values (such as negative claim frequencies), its application is only asymptotically justifiable, and major departures from the assumed normal distribution are possible for finite sample sizes.

The non-parametric bootstrapping method resamples only from values that have already appeared in the past and thus tends to underestimate extreme observations. However, this obviously limits the range of obtained parameters for non-parametric bootstrapping, a limitation that can be avoided by using parametric bootstrapping. The bootstrap method is applicable to arbitrary underlying distributions and estimators, and is therefore also applicable to any relevant non-life insurance model. However, the larger the original data set, the more time consuming the entire bootstrapping process. It is important to note that sufficiently large (observed) data sets often eliminate the necessity of considering parameter risk. We illustrate this aspect in an example in this section.

Finally, considering the Bayesian approach, the choice of a priori distribution is unclear. Unfortunately, there is no general analytical solution for posterior distributions, so we have to fall back on numerical methods, such as the Gibbs/ARMS sampling methods (see Gilks et al., 1994, 1995).

For illustration, we present the parameter distributions for our Poisson claim numbers sample set achieved by employing the three approaches. We performed 60,000 simulations. We illustrate the influence of data size on parameter risk and compare the parameter distributions from the original data set of 11 years'worth of history with 30-year sample achieved by adding another 19 observations. As can be seen, the ML estimates of the Poisson parameter (11 years: 1.45; 30 years: 1.50) differ slightly and the variance of the estimators decreases.

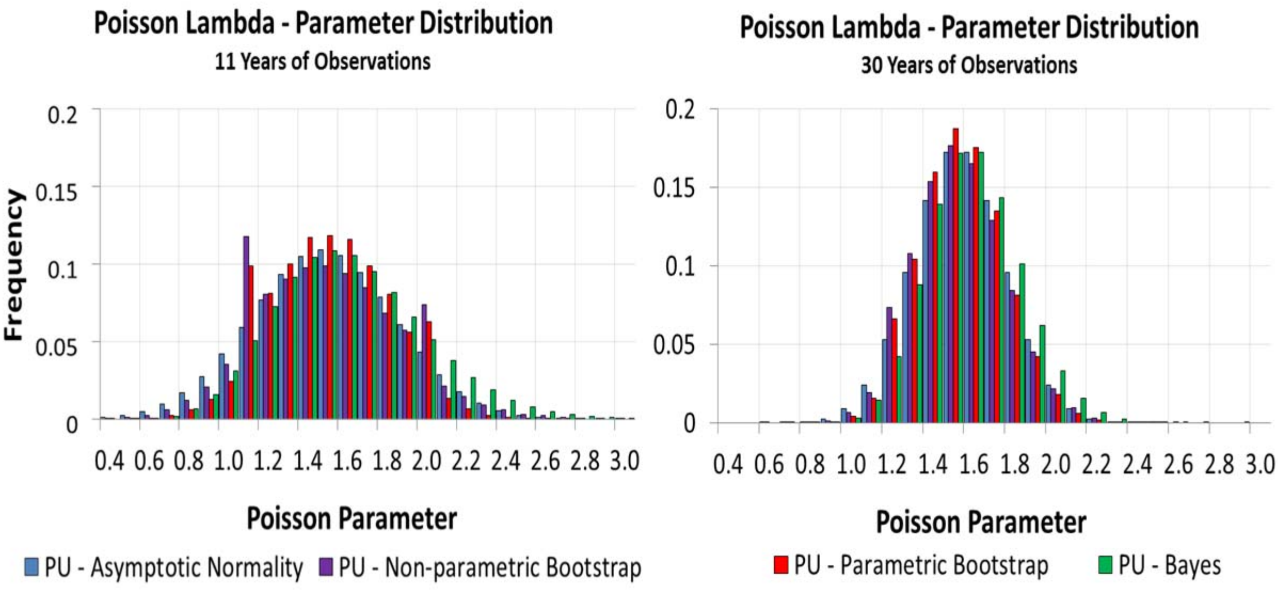

Figure 2: Frequencies for the Poisson parameter distributions underlying 11 and 30 years of observations

As shown in Figure 2 and Table 2, each approach results in a different parameter distribution. For the 11-year data set, the asymptotic normality approach leads to negative and hence non-feasible parameters. The parameter distributions with 30 years of observations are more concentrated around the estimated parameter than is the case for the 11-year data sample.

	11 Years				**30 Years**			
	Asymptotic Normality	Non-parametric Bootstrap	Parametric Bootstrap	Bayes	Asymptotic Normality	Non-parametric Bootstrap	Parametric Bootstrap	Bayes
Mean	1.45	1.45	1.45	1.55	1.50	1.50	1.50	1.53
Standard Deviation	0.36	0.30	0.36	0.37	0.22	0.21	0.22	0.23
Min	-0.12	0.27	0.18	0.42	0.53	0.63	0.60	0.74
Max	3.02	2.64	3.27	4.00	2.46	2.43	2.53	2.85
50th percentile	1.45	1.45	1.45	1.52	1.50	1.50	1.50	1.52
60th percentile	1.55	1.55	1.55	1.61	1.56	1.53	1.56	1.58
70th percentile	1.65	1.64	1.64	1.72	1.62	1.60	1.62	1.64
80th percentile	1.76	1.73	1.73	1.85	1.69	1.67	1.69	1.72
90th percentile	1.92	1.82	1.91	2.04	1.79	1.77	1.79	1.83
95th percentile	2.05	1.91	2.09	2.21	1.87	1.87	1.87	1.92

Table 2: Statistics of Poisson parameter distributions underlying 11 and 30 years of observations

In the next step we present the resulting predictive distributions for future claim numbers and compare it to the pure Poisson process distribution without parameter risk. Due to shortcomings of the non-parametric bootstrap approach, which tends to underestimate extreme observations, and of the asymptotic normality approach, which leads to negative, hence non-feasible, parameter values, we focus on the parametric bootstrap and the Bayesian approaches.

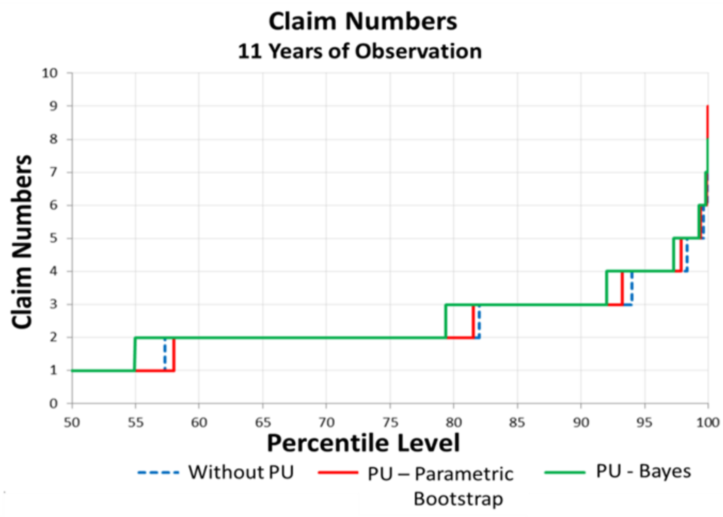

Figure 3: Percentile plot for prediction distributions of claim numbers for 11 years of observations

	Prediction distribution		
	Without Parameter Uncertainty	Parametric Bootstrap	Bayes
Mean	1.45	1.45	1.55
Standard Deviation	1.21	1.26	1.30

Table 3: Statistics for prediction distributions of claim numbers for 11 years of observations

Note that the predictive distributions have a higher variance than mean; thus we observe a higher probabilty for more extreme claim numbers (see Figure 3 and Table 3).

4. Application in a Non-Life Insurance Company

4.1 Model Framework and Assumptions

To illustrate the influence of parameter risk on internal model results and strategic decision making, we perform a simulation study using the multi-year model outlined in Section 2.3. We consider a monoliner for m=1 and m=5 subsequent accident years. The study was carried out using TW Igloo simulation software. Because of the disadvantages of the asymptotic normality and non-parametric bootstrap approaches (see Section 3.1), in this section we concentrate on the Bayesian and parametric bootstrap methods. For simplicity, we do not include premiums or costs, but focuses on cumulative claims for m years.

The constant premiums and costs yields a parallel shift in the overall distribution result, but this does not affect the parameter risk analyzed in this paper. We thus ignore premiums and costs in the following. Apart from modeling gross claims, this study illustrates the influence of parameter risk on the effect of risk reduction measures such as reinsurance. The reinsurance contract is applied to simulated gross claims, assuming a stop-loss contract on total claims by accident year. The following common assumptions are made regarding the process distributions (see Diers, 2007):

- Large claim losses X follow a Pareto (α) distribution ($\alpha>0$, Threshold $T>0$) with density:

$$f_X(x;\alpha;T) = \alpha \cdot \frac{T^\alpha}{x^{\alpha+1}}, x > T,$$

- Large claim numbers N follow a Poisson (λ) distribution, as introduced in Section 3.

- For attritional claims S^A, we assume a Gamma (α,β) distribution ($\alpha,\beta>0$) with density:

$$f_{S^A}(x;\alpha;\beta) = \frac{x^{\alpha-1}\exp\{-x/\beta\}}{\Gamma(\alpha)\beta^\alpha}, \quad x > 0.$$

The data set consists of fictive claim figures (11 accident years distinguishing between large and attritional claims) and is available upon request. Separation into attritional and large claims is based on a large loss threshold of T=1.2 million, resulting in 16 observed large claims above this threshold.

4.2 Results—Parameter Risk

The distribution parameters were estimated using the maximum likelihood method applied to 11 years' worth of data. We obtained:

- $\hat{\lambda}_N = 1.45$ for Poisson number of claims,

- $\hat{\alpha}_X = 1.56$ for Pareto large claim size,

- $\hat{\alpha}_{S^A} = 57.8$ and $\hat{\beta}_{S^A} = 1.33$ for the Gamma attritional claims.

As we are only analyzing the 11-year data set in this study, we omit the superscript indicating sample size. Assuming a uniform prior, the posterior distribution of the Pareto parameter α_X for a given data set $x^{(k)} = (x_1, \ldots, x_k)$ is a Gamma (α,β) distribution (see Fink, 1995) with

$$\alpha = m + 1 \quad \text{and} \quad \beta = \frac{1}{\sum_{i=1}^{k} \ln x_i - k \ln T}.$$

Even under a uniform prior, the two-variate posterior distribution for the Gamma parameter vector is complex, and we thus do not include it here. For details, the reader is again referred to Fink (1995).

Parameter Distributions

Sixty-thousand simulations were run to obtain the parameter and process distributions. The distributions for the Poisson parameter are discussed above in Section 3.

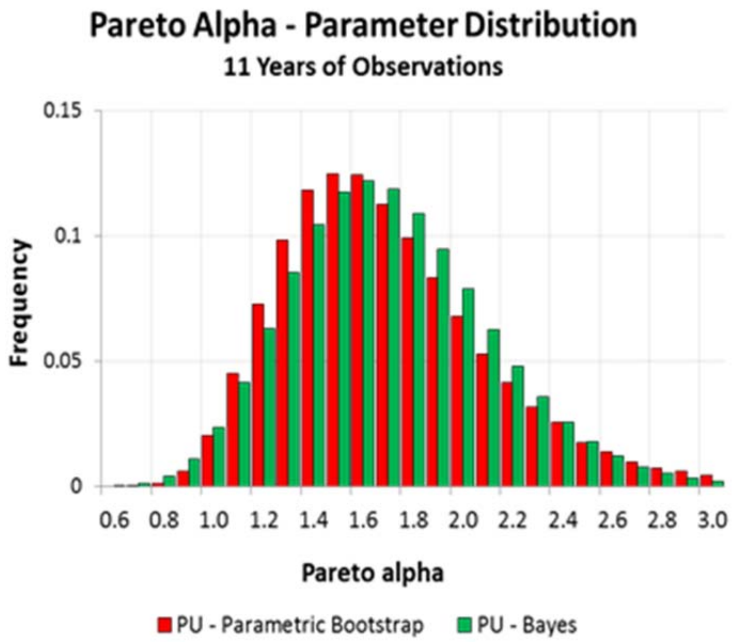

Figure 4: Pareto alpha parameter distribution

	Pareto Alpha	
	Parametric Bootstrap	Bayes
Mean	1.66	1.65
Standard Deviation	0.44	0.40
Min	0.67	0.48
Max	6.38	4.54
5th percentile	1.08	1.05
25th percentile	1.35	1.37
50th percentile	1.59	1.62
75th percentile	1.89	1.90
95th percentile	2.47	2.36

Table4: Statistics for Pareto alpha parameter distribution

In the Pareto large claim size model, it is important to note that the lower the parameter α, the thicker the tail of the Pareto distribution and hence the more risky the loss distribution. In the case of $\alpha<1$, we have an infinite mean model. As can be seen from Figure 4 and Table 4, the parameter estimate is far above 1, but around 5% of the parameters lie in the infinite mean. This might cause simulations with extreme single losses when sampling from the Pareto process distribution. Comparing the mean of the bootstrap parameter distribution to the initial estimate of alpha (1.56), we notice a small bias in the ML estimation.

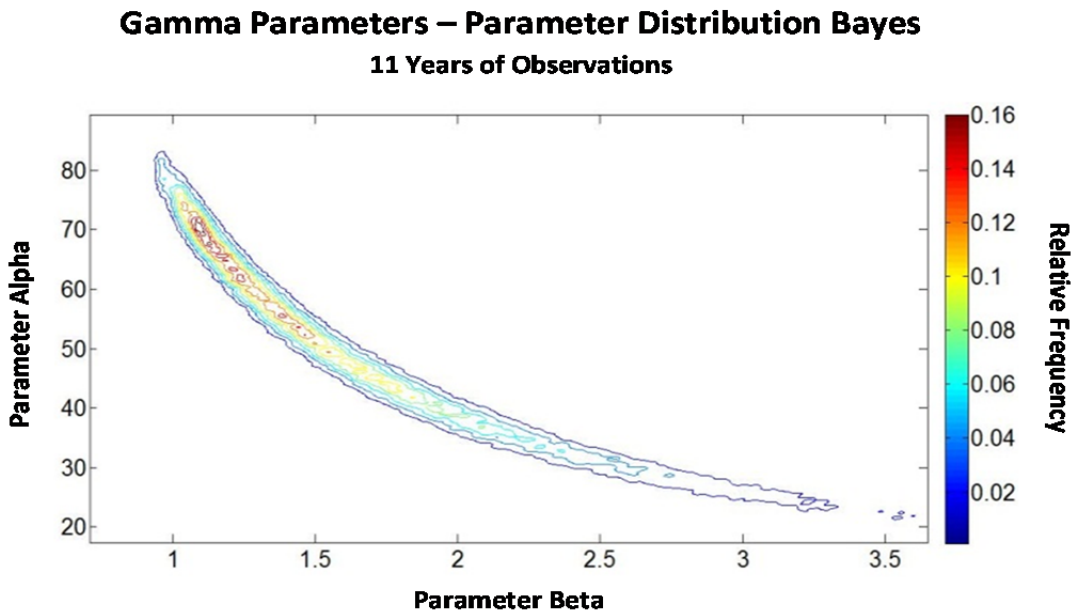

Figure 5: Scatter plot—Simulated Gamma parameters for attritional claims obtained from the Bayesian approach

Regarding the attritional losses, the ARMS method was used to generate the posterior parameter vector distribution for the Gamma distribution in the Bayesian approach (see Gilks et al., 1994, 1995). Figure 5 shows the simulated alpha and beta parameters (60,000 simulations). As can be seen from the figure, it preserves the negative dependency between alpha and beta parameters.

4.3 Results—Total Gross Claim Losses

The bulk of attritional claims and large claim losses are simulated independently and aggregated to the total gross loss.This procedure is repeated for each of the subsequent accident years; the total losses are path-wise aggregated.

Figure 6 reveals that incorporation of parameter risk leads to a higher probability of more extreme total losses, with the Bayesian approach even more extreme than the parametric bootstrapping case. Most especially for the multi-year case there is a significant distance between the loss distribution without parameter risk and the full predictive distributions (containing parameter risk). In this paper we limit ourselves to using only value-at-risk as a risk measure and employ a confidence level of 99.5%. Results for tail-value-at-risk, other percentile levels, and comparisons basedon 5 or 30 years of data are available on request.

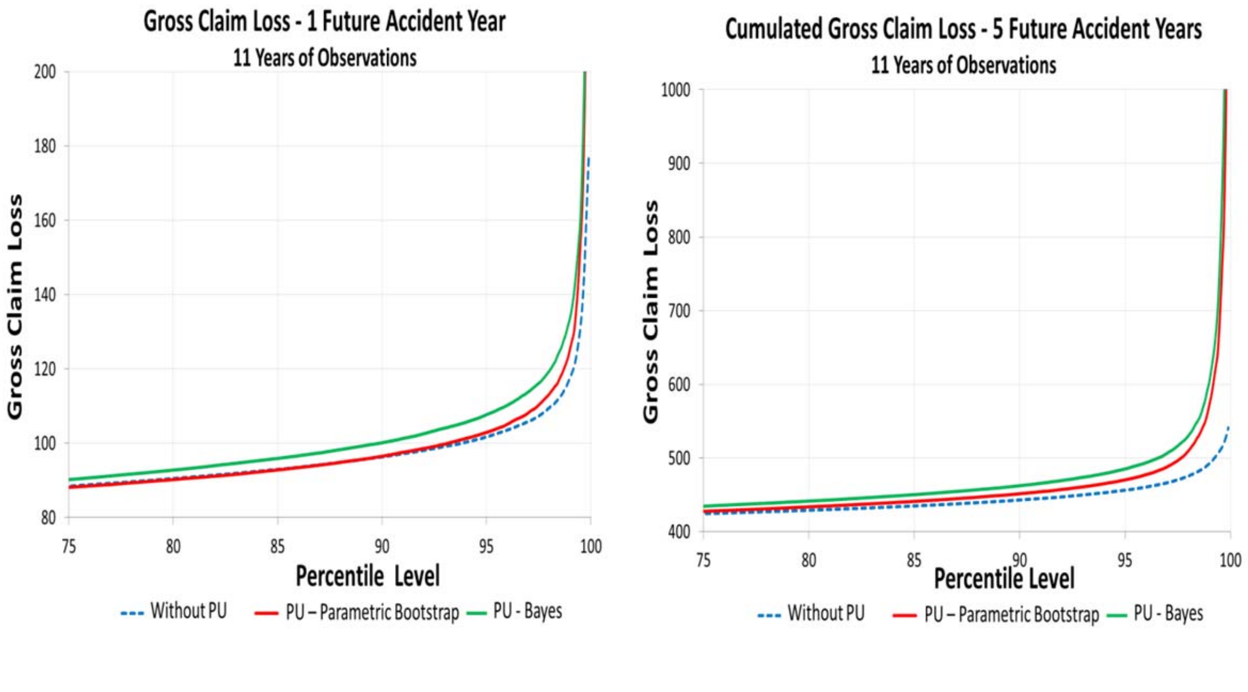

Figure 6: Percentile plot of the total gross claim loss for one and five accident years

Number of future accident years m		Without parameter risk	Bayes	Parametric bootstrap
1	Mean	81.61	85.02	83.97
	Standard deviation	12.71	230.58	296.12
	$VaR_{99.5}$	131.22	160.49	153.4
	$VaR_{99.5}$ - Mean	49.61	75.47	69.7
5	Mean	408.06	425.92	416.39
	Standard deviation	28.37	603.75	418.63
	$VaR_{99.5}$	509.94	766.87	690.45
	$VaR_{99.5}$ - Mean	101.88	340.95	274.07

Table5: Results for total gross claim losses $S_{[t+1,t+m]}$

The significant increase of the standard deviation mainly results from extreme large claims simulations, caused by Pareto parameters in the infinite mean region. The difference between value-at-risk (VaR) 99.5% and mean gross claim losses in Table 5 ($VaR_{99.5}$- Mean) can be interpreted as the amount necessary to cover adverse deviations of claims from the expectation value and is an alternative approach for measuring premium risk. We observe that in the one-year case it increases by 52% (Bayes) and 40% (parametric bootstrap); for the five-year case, it increases by 235% (Bayes) and 169% (parametric bootstrap).

We measure the diversification effect by taking the difference between $5 \cdot (VaR_{99.5}\text{-Mean})_{m=1}$ and $(VaR_{99.5}\text{-Mean})_{m=5}$. The diversification effect decreases significantly from 59% (without parameter risk) to 10% (Bayes) and 21% (parametric bootstrap), indicating that there is a high degree of dependency between the accident years via common modeling of parameter risk. We thus demonstrate that including parameter risk can substantially increase gross claim size and reduce diversification benefits in multi-year modeling.

4.4 Results—Reinsurance Results

Reinsurance is an essential means for non-life insurers to lower their capital charge by transferring premium risk. To illustrate how reinsurance decisions are affected by parameter risk, we analyze a stop-loss reinsurance contract with a €200 million limit (L) and a retention of €90 million for total claim losses (E) per accident year. Reinsurance recoveries per single future accident year i = $t+1,\ldots,t+m$ can be calculated pathwise in a simulation environment as follows:

$$R_{t+i} := \min(\max(S_{t+i} - E;0); L) .$$

We define cumulated recoveries for all future accident years as $R_{[t+1,t+m]} := \sum_{i=1}^{m} R_{t+i}$.

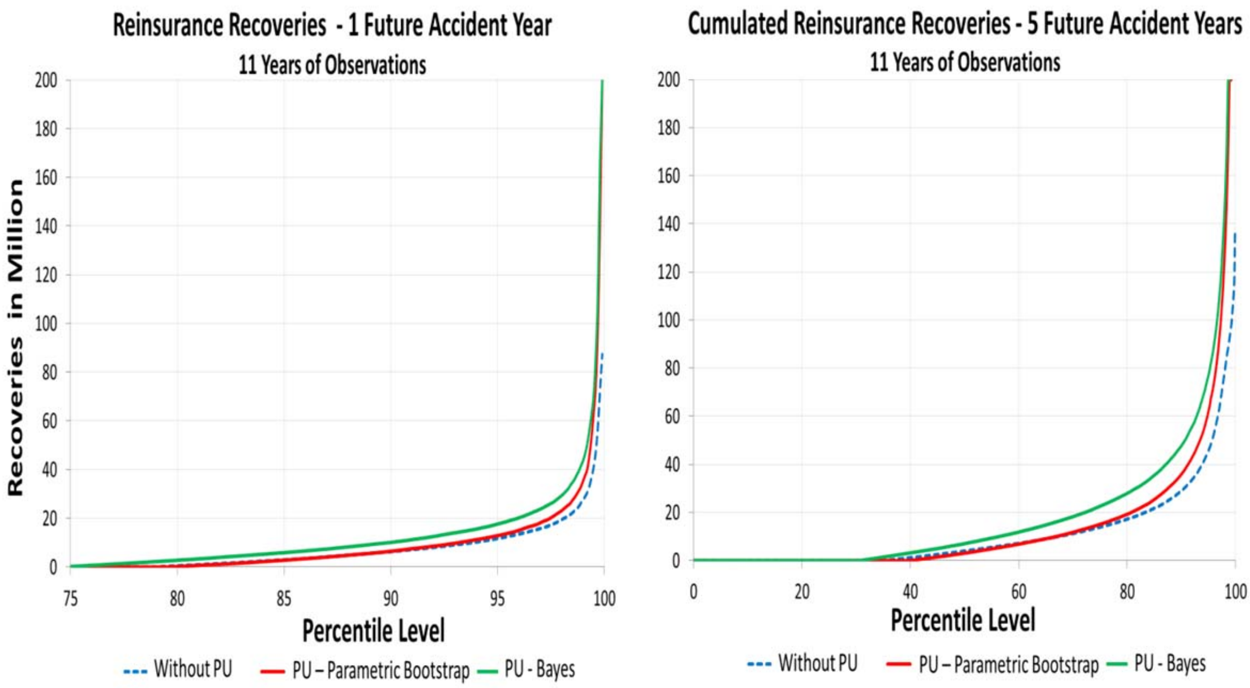

Figure7: Percentile plot of reinsurance recoveries

Number of future accident years m		Without parameter risk	Bayes	Parametric bootstrap
1	Mean	1.93	3.35	2.44
	Standard deviation	6.65	12.52	11.30
	$VaR_{99.5}$	41.22	70.49	63.44
5	Mean	10.95	19.30	14.52

	Standard deviation	18.51	36.14	32.40
	$VaR_{99.5}$	105.79	222.26	211.27

Table 6: Results for reinsurance recoveries $R_{[t+1,t+m]}$

If we compare the percentile plots of reinsurance recoveries with and without parameter risk, it is clear that the reinsurance company, too, is subject to parameter risk (Figure 7 and Table 6). This may have an important impact on the structure of reinsurance programs (especially regarding parameters such as retention and limits) and the pricing of reinsurance contracts. Figure 7 illustrates this effect compared to net losses.

We conclude that pricing reinsurance contracts via a value-at-risk approach or based on a standard deviation principle will lead to substantially different results when parameter risk is taken into consideation. This degree of difference will depend on the size of the underlying observed data sets. Sufficiently large data sets (data on more than 50 claims) often eliminate the need to consider parameter risk, but such large data sets are very rarely available.

5. Conclusion

In this paper, we present a multi-year model framework for embedding parameter risk in non-life premium risk in enterprise risk management. We illustrate different modeling approaches for parameter risk itself, namely, the asymptotic normality approach, bootstrap methods, and Bayesian methods, identifying the advantages and disadvantages of each, with special focus on their practicality.

A simulation study was conducted using the Bayesian method and parametric bootstrapping in a multi-year enterprise risk management framework. The results clearly show that parameter risk in premium risk may have a substantial effect on the company's risk situation, and is a risk that needs to be sufficiently taken into consideration. The study also demonstrates remarkable differences between including and excluding parameter risk. Ignoring parameter risk can lead to a substantial underestimation of risk capital requirements, depending on the size of the underlying (observed) data sets.

This finding that ignoring parameter risk may severly underestimate capital requirements is even more crucial when it comes to multi-year model projections, as the diversification effect between the individual accident years including parameter risk is significantly lower than when excluding parameter risk. Finally, considering risk transfer with the help of a stop-loss reinsurance contract shows that it is reinsurance companies that bear most of the parameter risk, a finding that should be of high interest to those designing and pricing reinsurance contracts. Indeed, we believe that although our findings will be of interest to academics, they will be of greatest value to real-world insurance and reinsurance businesses. Therefore, we recommend including the modeling approaches presented here in insurers' internal risk models.

With regard to further development and research opportunities, modifications to the basic data are often needed in the non-life sector (as-if transformation, as with settlement, in the long-tail divisions, detrending, etc.) before these values can be used for parameterization. These adjustments are intended to ensure that the elements from the original data set represent values estimated. In practice, this leads to other sources of error that affect parameter estimates, but that cannot be included in the model using the approaches presented. Further analysis may provide clarity as to the scale of error involved.

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