Institute of Insurance Economics

University of St.Gallen

THE INFLUENCE OF STOCHASTIC INTEREST RATES ON THE VALUATION OF PREMIUM PAYMENT OPTIONS IN PARTICIPATING LIFE INSURANCE CONTRACTS

HSIAOYIN CHANG
HATO SCHMEISER

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Hsiaoyin Chang* and Hato Schmeiser†

Institute of Insurance Economics
University of St.Gallen

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Abstract

Life insurance contracts typically possess various embedded options. In this paper, we particularly focus on options with early exercise features such as paid-up options, resumption options, surrender options and combinations of these. We investigate how the option values change under different parameters and exercise strategies. In contrast to the existing literature, which has shown that the values of premium payment options are rather small under a deterministic term structure, we demonstrate that the situation changes dramatically whenever stochastic interest rates are introduced.

1 Introduction

Life insurance contracts are typically offered with various embedded options. In this paper, we focus in particular on premium payment options with early exercise features, which can be found in essentially any life insurance contract with regular premium payments. A paid-up option allows policyholders to stop premium payments while the main contract continues with adjusted benefits. Resumption options allow policyholders to resume payments after the paid-up option has been exercised (again, benefits will be adjusted accordingly). With a surrender option, policyholders can terminate their contract and receive a surrender amount before maturity. With a combined paid-up and surrender option, policyholders may surrender their policy with or without previously exercising the paid-up option.

In the current low-interest rate environment, insurance companies are particularly struggling with the high long-term interest guarantees which they previously provided to their policyholders. The situation for the insurer can be even more problematic, as policyholders tend to exercise their surrender or paid-up options once the interest rate rebounds (cf. Feodoria and Förstemann (2015)). Therefore, if no proper risk management has taken place and hence options are not

*Email address:hsiaoyin.chang@unisg.ch
†Email address: hato.schmeiser@unisg.ch

priced adequately, insurance companies may encounter severe difficulties (cf. the cases of Equitable Life in 2000 or The Hartford in 2009). In addition, current solvency regulation schemes such as Solvency II or the Swiss Solvency Test require insurers to consider lapse risk and provide proper risk management and equity capital for options provided to their customers. Proper models for the valuation of premium payments options and the related risk assessment are thus essential for life insurance companies and should be conducted with care.

Similar to Bermuda or American options, valuation of premium payment options with early-exercise features is often complicated, because assumptions must be made about policyholders' behaviors. Andreatta and Corradin (2003) compare option values using the least squares Monte Carlo method (LSMC), suggested in Longstaff and Schwartz (2001), and the recursive tree binomial approach described in Bacinello (2003). They conclude that the two approaches are similarly accurate, while LSMC requires less CPU time. Bauer, Bergmann, and Kiesel (2010) built a general model involving the existing models for tackling these options. They compare three numerical valuation methods: nested Monte Carlo, partial (integro-)differential equations (PDE approach), and LSMC. Again, LSMC was found to be superior because of its efficiency. In addition, they show that - based on a numerical example - the value of surrender options are rather small. However, the authors make the point that the situation may change for specific parameter combinations like increasing interest rates.

Kling, Russ, and Schmeiser (2006), Gatzert and Schmeiser (2008), and Schmeiser and Wagner (2011) use geometric Brownian motion for assets and a deterministic interest rate to value joint premium payment options. They analyze a paid-up option, a joint option of paid-up and resumption, a surrender option, and a joint option of paid-up and surrender. Based on a fair pricing concept (insurance contracts have a net present value of zero), they treat option values as the difference between the contract value with and without premium payment options. In particular, if the options are exercised at their maximum level, the provider may face severe risk (cf. Gatzert and Schmeiser (2008)). However, as Kling et al. (2006), Reuß, Ruß, and Wieland (2016), and Gatzert and Schmeiser (2008) point out, this strategy is not feasible from the policyholders' point of view. Schmeiser and Wagner (2011) value premium payment options with an optimal stopping strategy, as discussed in Kling et al. (2006) and Andersen (1999). The strategy is based on deterministic interest rates; under this assumption, the value of premium payment options is rather small (cf. Schmeiser and Wagner (2011)).

However, Kuo, Tsai, and Chen (2003) showed in their empirical findings that interest rate changes strongly influence policyholders' option exercising behaviors. Given this line of reasoning and based on the general framework introduced in Schmeiser and Wagner (2011), we show that the influence of stochastic interest rates on the valuation of premium payment options is enormous. This fact must be considered to prevent underpricing of life insurance contracts. In addition, only risk-adequate pricing allows insurers to provide adequate risk management measures such as equity capital to always ensure payments to policyholders. To obtain numerical results, we use an adjusted LSMC setup. We assume two stochastic risk sources (assets and interest rates) and introduce rational policyholders who follow an optimal exercise strategy with respect to the embedded options.

The remainder of this paper is organized as follows: Section 2 introduces the contract framework from Schmeiser and Wagner (2011) and extends the setup by using the stochastic Vasicek model (cf. Vasicek (1977)). Section 3 analyzes option values based on both a rational and feasible exercise strategy and assuming options are maximally exercised. Section 4 provides several numerical results. Section 5 discusses the economic implications of our findings and concludes the paper.

2 The Model Framework

The basic contract includes two standard options: a guaranteed yearly interest rate (g) and a surplus participation (with participation rate α). We then extend the basic contract with four different premium payment options: a paid-up option, $\vartheta^{\mathcal{P}}$; a combined paid-up and resumption option, $\vartheta^{\mathcal{PR}}$; a surrender option, $\vartheta^{\mathcal{S}}$; and a combined paid-up and surrender option, $\vartheta^{\mathcal{PS}}$.[1] These premium payment options are assumed to only be exercised at the end of each year, given that the main basic contract is still in force (i.e., at the end of year t, policyholders must still be alive and the relevant options not yet exercised). We assume the insurer faces no default risk and hence legitimate payments to the policyholders can always be achieved by the insurer. In other words, the insurer can hedge out all the risk from both the basic contract options and premium payment options.

2.1 The Basic Contract (Π)

We start with a basic life insurance endowment contract with a duration of T years and time index $t = 1...T$. Let ${}_tp_x$ be the probability that a policyholder aged x years survives the next t years, while ${}_tq_x$ (=$1 - {}_tp_x$) represents the probability of death over the next t years. Following actuarial practice, we assume mortality risk is negligible. More precisely, it is assumed that mortality risk is uncorrelated to financial risk sources and hence has a pure unsystematic nature.

Annual premium payments, B_t, are paid by the policyholder at the beginning of t provided the policyholder is alive at the end of $t-1$. As is common in insurance practice, premium payments are constant in time, i.e., $B_t = B$. The present value (PV) of premium payments can be written as $B \sum_{t=0}^{T-1} {}_tp_x(1+r)^{-t}$, where r is the technical discount rate.

Benefit payments provided to the policyholder include death benefits and survival benefits. If the policyholder dies during year t, death benefits are payable at the end of year t. The death benefits are constant and the PV can be formalized as $\gamma \sum_{t=0}^{T-1} {}_tp_x q_{x+t}(1+r)^{-(t+1)}$. Survival benefits are paid out at T if the policyholder survives when the contract matures. γ is the minimum amount of the survival benefits (guaranteed survival benefit) provided to the policyholder. The PV of the guaranteed survival benefit can be written as $\gamma {}_Tp_x(1+r)^{-T}$.

According to the actuarial equivalence principle, the PV of the premium payments and that of the death and survival benefits should be identical. Hence, γ can be derived with a fixed premium payment amount B. In order to be on the safe side, we discount the premium and benefit payments using the interest guarantee rate, g (cf. Linnemann (2003)). The relationship between the premium payments and the benefits is shown via the following equation:

$$B \sum_{t=0}^{T-1} {}_tp_x(1+g)^{-t} = \gamma\left(\sum_{t=0}^{T-1} {}_tp_x q_{x+t}(1+g)^{-(t+1)} + {}_T p_x(1+g)^{-T}\right) \tag{1}$$

[1]We owe our definitions of these basic contract forms to the paper by Schmeiser and Wagner (2011).

Hence, γ is given by

$$\gamma = \frac{B \sum_{t=0}^{T-1} {}_tp_x (1+g)^{-t}}{\sum_{t=0}^{T-1} {}_tp_x q_{x+t} (1+g)^{-(t+1)} + {}_T p_x (1+g)^{-T}} \tag{2}$$

Note that for the survival benefit, γ only represents the minimum benefit given by the guaranteed interest rate. The actual amount of the survival benefit depends on the policy's accumulated assets, A_T, including both the guarantee option and a surplus participation. To calculate this policy's accumulated asset, A_t, we separate the annual premium payment, B, into two parts, denoted by B_t^R and B_t^A. B_t^R as $q_{x+t-1} \max(\gamma - A_{t-1}, 0)$ is used to pay the difference between the death benefits and the policy's asset accumulated by the end of the previous year (A_{t-1}). The remainder, B_t^A serves as the savings premium and becomes part of the policy's accumulated asset account for the coming year, t:

$$\begin{aligned} B &= B_t^R + B_t^A \\ B_t^A &= B - q_{x+t-1} \max(\gamma - A_{t-1}, 0) \end{aligned} \tag{3}$$

At the beginning of t, the policy's accumulated asset contains two parts: the accumulated amount at the end of the previous year, A_{t-1}, and the annual savings premium, B_t^A, collected at the beginning of t. With both the guarantee and surplus options, the accumulated assets earn an annual return at the guaranteed interest rate or an annual surplus rate, whichever is greater. The annual surplus rate is a fraction, α, of the annual insurer's investment result at t, i.e., $S_t/S_{t-1} - 1$. Hence, α serves as a participation rate. The development of the policy's assets over time can be formally written as

$$A_t = (A_{t-1} + {}_{t-1}p_x B_t^A)(\max(g, \alpha(S_t/S_{t-1} - 1)) + 1) \tag{4}$$

with $A_0 = 0$

The policy's asset is subject to investment risk, which includes two risk sources, the interest rate risk and the asset risk. The interest rate, r, evolves according to the one-factor Vasicek model (cf. Vasicek (1977)):

$$dr_t = \kappa(\theta - r_t)dt + \sigma_I dZ^{\mathbb{P}} \tag{5}$$

Here, $Z^{\mathbb{P}}$ is a Wiener process on a probability space $(\Omega, \phi, \mathbb{P})$. To capture the interest rate risk, σ_I determines how much randomness of Z is acquired in the model. κ and θ are positive constants representing the speed of reversion and the long-term mean, respectively. A constant market price of risk, λ, is introduced to transfer the model into the risk-neutral probability space. If the market participants are risk averse, we have $\lambda < 0$. Under the risk-neutral measure, $\mathbb{Q}$, the interest spot rate process given in equation (5) changes to

$$dr_t = \kappa(\theta - \frac{\sigma_I \lambda}{\kappa} - r_t)dt + \sigma_I dZ^{\mathbb{Q}}, \tag{6}$$

where $Z^{\mathbb{Q}}$ denotes the Wiener process under the risk-neutral measure, $\mathbb{Q}$. The solution of the Vasicek model for one period return can be derived as

$$r_t = e^{(-\kappa \Delta t)} + (\theta - \frac{\sigma_I \lambda}{\kappa})(1 - e^{-\kappa \Delta t}) + \frac{\sigma_I}{\sqrt{2\kappa}} \sqrt{1 - e^{-2\kappa \Delta t}} Z_t^{\mathbb{Q}} \tag{7}$$

For asset risk, we assume the policy's asset follows a geometric Brownian motion (μ and σ_s) with stochastic interest rate derived via equation (7). For the geometric Brownian motion, we have:

$$d(\ln S_t) = (\mu - \sigma_s^2/2)t + \sigma_s dW$$

Under the risk-neutral measure, $\mathbb{Q}$, and combined with a stochastic interest rate, the deterministic drift for the asset risk changes to the stochastic spot rate and hence leads to

$$\begin{aligned} dS_t &= r_t S_t dt + \sigma_s S_t dW^{\mathbb{Q}} \\ \ln \frac{S_t}{S_{t-1}} &= r_t - \sigma_s/2 + \sigma_s(\rho Z^{\mathbb{Q}} + \sqrt{1-\rho^2} W^{\mathbb{Q}}) \end{aligned} \tag{8}$$

In this context, $W^{\mathbb{Q}}$ represents a second Wiener process under the risk-neutral measure $\mathbb{Q}$. σ_s captures the investment risk, which relates to both the asset risk and the interest risk. ρ indicates the correlation coefficient between the interest rate risk and the asset risk.

The contract value at t, denoted by Π_t, is the difference between two cash flows valued at t: the benefit paid to the policyholder and the premium paid by the policyholder to the insurer. The net present value (NPV) of the contract at $t = 0$ is the difference between the PV of these two cash flows:

$$\Pi_0 = E^{\mathbb{Q}}[(\gamma \sum_{t=0}^{T-1} {}_t p_x q_{x+t} \delta_{t+1} + A_T \delta_T - B \sum_{t=0}^{T-1} {}_t p_x \delta_t)] \tag{9}$$

$\delta_t = \Pi_0^t((1 + r_t))^{(-1)}$ is a discounting factor under the risk-neutral measure at the end of period t back to the beginning of the contract. We call a contract fair whenever its NPV is zero ($\Pi_0 = 0$).

For different parameters, we aim to derive their respective participation rate, α (with $0 \leq \alpha \leq 1$), that leads to a fair condition for policyholders and equity holders (Π_0 must be zero).

In what follows, the value of premium payment options is derived by calculating the PV difference between the basic contract (with the investment guarantee and surplus option only) and the basic contract plus premium payment options. The premium payment options considered in this paper are: paid-up option, combined paid-up and resumption option, surrender option, and combined paid-up and surrender option.

g	guaranteed interest rate
r_t	stochastic annual spot rate for year t
δ_t	stochastic discount factor for year t back to year 0
α	participation rate ($0 \leq \alpha \leq 1$)
Π_t	basic contract value at year t
B_t	constant premium payment, paid at the beginning of year t
	$(B_t = B = B_t^A + B_t^R)$
B_t^R	term life premium
B_t^A	saving premium
γ	constant death benefit paid at the end of the year
A_t	policy's accumulated asset at year t
PV	present value (at year 0)
ϑ	present value of the option

Table 1: Summary of basic contract notation

2.2 Contract with a Paid-up Option ($\vartheta_\tau^{\mathcal{P}}$)

In this section, we extend the basic contract form with an additional paid-up option. Once the paid-up option is exercised, policyholders stop premium payments while the contract continues with adjusted benefits. Note that, in this scenario, policyholders cannot resume payments once the paid-up option has been exercised.

In what follows, $\gamma_\tau^{\mathcal{P}}$ denotes the adjusted benefits when the paid-up option is exercised at $t = \tau$, $\tau = 1...T$. When $\tau = T$, the option has expired and thus its value is zero. The benefit, $\gamma_\tau^{\mathcal{P}}$, depends on the accumulated assets at τ and the survival probabilities of the insured with age $x + \tau$. In formal terms, this gives:

$$\gamma_\tau^{\mathcal{P}} = \frac{A_\tau}{\sum_{t=\tau}^{T-1} {}_{t-\tau}p_{x+\tau} q_{x+t} (1+g)^{-(t-\tau+1)} + {}_{T-\tau}\, p_{x+\tau}(1+g)^{-(T-\tau)}} \tag{10}$$

The adjusted final survival benefit, $A^{\mathcal{P}}_{T,\tau}$, is based on the sum of $A^{\mathcal{P}}_{t-1,\tau}$ and the savings premium, $B^A_t = 0 - B^R_t = -q_{x+t-1}\max(\gamma - A_{t-1}, 0)$:

$$A^{\mathcal{P}}_{t,\tau} = (A^{\mathcal{P}}_{t-1,\tau} - q_{x+t-1}\max(\gamma - A^{\mathcal{P}}_{t-1,\tau}, 0))(\max(g, \alpha(S_t/S_{t-1} - 1)) + 1), \tag{11}$$

with $A^{\mathcal{P}}_{\tau,\tau} = A_\tau$.

Using the risk-neutral valuation technique, the PV of the contract, $\Pi^{\mathcal{P}}_{0,\tau}$, is thus given by:

$$\Pi^{\mathcal{P}}_{0,\tau} = E^{\mathbb{Q}}[\gamma \sum_{t=0}^{\tau-1} {}_tp_x q_{x+t}\delta_{t+1} + \gamma^{\mathcal{P}}_\tau \sum_{t=\tau}^{T-1} {}_tp_x q_{x+t}\delta_t + A^{\mathcal{P}}_{T,\tau}\delta_T - B\sum_{t=0}^{\tau-1} {}_tp_x\delta_t] \tag{12}$$

The paid-up option value, $\vartheta^{\mathcal{P}}_\tau$, can finally be determined by the difference between the PVs of the contracts with and without a paid-up option:

$$\begin{aligned} \vartheta^{\mathcal{P}}_\tau &= \Pi^{\mathcal{P}}_{0,\tau} - \Pi_0 \\ &= E^{\mathbb{Q}}[(\gamma^{\mathcal{P}}_\tau - \gamma)\sum_{t=\tau}^{T-1} {}_tp_x q_{x+t}\delta_{t+1} + (A^{\mathcal{P}}_{T,\tau} - A_T)\delta_T + B\sum_{t=\tau}^{T} {}_tp_x\delta_t] \end{aligned} \tag{13}$$

2.3 Contract with a Combined Paid-up and Resumption Option ($\vartheta^{\mathcal{PR}}_{\tau,\nu}$)

In this section, we add a resumption option to the basic contract with a paid-up option alone (described in the previous section). The resumption option allows policyholders to resume premium payments after the paid-up option is exercised. Let $\gamma^{\mathcal{PR}}_{\tau,\nu}$ denote the adjusted benefits when exercising the paid-up option at τ and exercising the resumption option at ν, with $\nu = \tau + 1...T$ for $\tau < T$ and $\nu = T$ when $\tau = T$. For $\nu = T$, the resumption option has expired without being exercised. The adjusted benefits, $\gamma^{\mathcal{PR}}_{\tau,\nu}$, are thus given by:

$$\gamma^{\mathcal{PR}}_{\tau,\nu} = \frac{A^{\mathcal{P}}_{\nu,\tau} + B\sum_{t=\nu}^{T-1} {}_{t-\nu}p_{x+\nu}(1+g)^{-(t-\nu)}}{\sum_{t=\nu}^{T-1} {}_{t-\nu}p_{x+\nu}q_{x+t}(1+g)^{-(t-\nu+1)} + {}_{T-\nu}\, p_{x+\nu}(1+g)^{-(T-\nu)}} \tag{14}$$

After exercising the resumption option, the policyholder resumes premium payments, B, into the contract. Hence, the accumulated assets, $A^{\mathcal{PR}}_{T,\nu,\tau}$, are given by

$$\begin{aligned} A^{\mathcal{PR}}_{t,\nu,\tau} &= (A^{\mathcal{PR}}_{t-1,\nu,\tau} + {}_{t-1}\, p_x B^A_t)(\max(g, \alpha(S_t/S_{t-1} - 1)) + 1) \\ B^A_t &= B - B^R_t = B - q_{x+t-1}\max(\gamma - A_{t-1}, 0), \end{aligned} \tag{15}$$

with $A^{\mathcal{PR}}_{\nu,\nu,\tau} = A^{\mathcal{P}}_{\nu,\tau}$.

The PV of the contract with paid-up option exercised at τ and resumed at ν can be formalized as

$$\begin{aligned}\Pi^{\mathcal{PR}}_{0,\nu,\tau} =& E^{\mathbb{Q}}[\gamma \sum_{t=0}^{\tau-1} {}_tp_x q_{x+t}\delta_{t+1} + \gamma^{\mathcal{P}} \sum_{t=\tau}^{\nu-1} {}_tp_x q_{x+t}\delta_{t+1} + \\ &\gamma^{\mathcal{PR}} \sum_{t=\nu}^{T-1} {}_tp_x q_{x+t}\delta_{t+1} + A^{\mathcal{PR}}_{T,\nu,\tau}\delta_T - B\sum_{t=0}^{\tau-1} {}_tp_x\delta_t - B\sum_{t=\nu}^{T-1} {}_tp_x\delta_t]\end{aligned} \tag{16}$$

The value of the combined paid-up and resumption option is given by the difference between the PVs of the contract with the combined option and the basic contract without the combined option:

$$\begin{aligned}\vartheta^{\mathcal{PR}}_{\tau,\nu} &= \Pi^{\mathcal{PR}}_{0,\nu,\tau} - \Pi_0 \\ &= E^{\mathbb{Q}}[(\gamma^{\mathcal{P}} - \gamma)\sum_{t=\tau}^{\nu-1} {}_tp_x q_{x+t}\delta_{t+1} \\ &+ (\gamma^{\mathcal{PR}} - \gamma)\sum_{t=\nu}^{T-1} {}_tp_x q_{x+t}\delta_{t+1} + (A^{\mathcal{PR}}_{T,\nu,\tau} - A_T)\delta_T + B\sum_{t=\tau}^{\nu-1} {}_tp_x\delta_t]\end{aligned} \tag{17}$$

2.4 Contract with a Surrender Option ($\vartheta^{\mathcal{S}}_{\theta}$)

A surrender option allows the policyholder to terminate the policy and receive a surrender value before the agreed end of maturity. This option can be exercised at a specific point in time, θ, with $\theta = 1...T$. If $\theta = T$, the surrender option has expired and thus its value is zero. The surrender value is assumed to equal the policy's accumulated assets at θ, denoted by A_θ.

The PV of the basic contract including a surrender option exercised at θ is denoted by $\Pi^{\mathcal{S}}_{0,\theta}$ and can be calculated as:

$$\Pi^{\mathcal{S}}_{0,\theta} = E^{\mathbb{Q}}[\gamma \sum_{t=0}^{\theta-1} {}_tp_x q_{x+t}\delta_{t+1} + A_\theta\delta_\theta - B\sum_{t=0}^{\theta-1} {}_tp_x\delta_t] \tag{18}$$

The surrender option value, $\vartheta^{\mathcal{S}}_{\theta}$, is the difference between the PV of the basic contract with and without the surrender option:

$$
\begin{aligned}
\vartheta_\theta^{\mathcal{S}} &= \Pi_{0,\theta}^{\mathcal{S}} - \Pi_0 \\
&= E^{\mathbb{Q}}[-\gamma \sum_{t=\theta}^{T} {}_tp_x q_{x+t}\delta_{t+1} + A_\theta\delta_\theta - A_T\delta_T + B\sum_{t=\theta}^{T} {}_tp_x\delta_t]
\end{aligned} \tag{19}
$$

2.5 Contract with a Combined Paid-up and Surrender Option ($\vartheta_{\theta,\tau}^{\mathcal{PS}}$)

Typical life insurance endowment policies allow policyholders to make the contracts paid-up and later surrender the policies before they mature. Hence, in this section we introduce a combined paid-up and surrender option. Besides exercising each option individually, policyholders may also exercise the paid-up option first and surrender the contract later (but not vice versa). When exercising the surrender option at θ after exercising the paid-up option at τ, policyholders receive the surrender value as the accumulated asset amount $A_{\theta,\tau}^{\mathcal{P}}$ with $\tau = 0...T-1$ and $\theta = \tau+1...T$. If $\tau = 0$ or $\theta = T$, the respective option has not been exercised. The PV of the contract with both paid-up and surrender options is thus given by

$$
\Pi_{0,\theta,\tau}^{\mathcal{PS}} = \begin{cases}
E^{\mathbb{Q}}[\gamma\sum_{t=0}^{\tau-1} {}_tp_x q_{x+t}\delta_{t+1} + \gamma^{\mathcal{P}}\sum_{t=\tau}^{\theta-1} {}_tp_x q_{x+t}\delta_{t+1} + \\
\quad A_{\theta,\tau}^{\mathcal{P}}\delta_\theta - B\sum_{t=0}^{\tau-1} {}_tp_x\delta_t], \quad T > \theta > \tau \\
\quad \text{if exercising both paid-up and surrender options} \\
\\
\Pi_{0,\tau}^{\mathcal{P}}, \text{if exercising paid-up option only}(\theta = T) \\
\\
\Pi_{0,\theta}^{\mathcal{S}}, \text{if exercising surrender option only}(\tau = 0)
\end{cases} \tag{20}
$$

The value of the combined paid-up and surrender option can be written as:

$$
\vartheta_{\theta,\tau}^{\mathcal{PS}} = \Pi_{0,\theta,\tau}^{\mathcal{PS}} - \Pi_0 \tag{21}
$$

3 Valuation of Premium Payment Options

In life insurance contracts, the assumed policyholder's exercise strategy is central when valuing embedded premium payment options. We begin by calculating the upper limit of the premium payment options for any possible exercise strategy. Since such a valuation uses information which is not accessible in a neoclassical finance setting and hence not a feasible strategy for policyholders, we follow this with an LSMC (least-squares Monte Carlo) strategy as an approximation of an optimal exercise approach and as the basis for the value of the premium payment options.

3.1 Deriving an Upper Limit ($^{UP}\vartheta$)

Kling et al. (2006), Gatzert and Schmeiser (2008), and Schmeiser and Wagner (2011) discuss calculating an upper limit for premium payment options and its economical interpretation in detail. Assuming policyholders know future developments, the premium payment option would be exercised at its maximum value for the whole contract period. In formal terms, we have:

$$^{UP}\vartheta = \frac{1}{N}\sum_{n=1}^{N}(\max(^{n}\vartheta_t, 0)) \qquad t = 1...T-1, \tag{22}$$

where $^{n}\vartheta_t$ denotes the different option values if exercised at t for the n^{th} simulation path, and $\max(^{n}\vartheta_t, 0)$ denotes the maximum amount during the whole contract period for the n^{th} path. Policyholders do not exercise these options if their value is negative for the whole contract period. The upper limit of the options can also be referred to as the PV given perfect information. Although perfect information is not feasible in practice, the concept still provides useful insight as it shows the upper bound of the option – or maximum loss from the insurer's viewpoint – under any conceivable exercise strategy while assuming no parameter or model risk occurs.

3.2 Option Valuation via the Least-squares Monte Carlo Strategy ($^{LSMC}\vartheta$)

The LSMC method was first presented by Longstaff and Schwartz (2001) to price American options. It has been use to value the surrender option and similar premium payment options in life insurance contracts (cf. Andreatta and Corradin (2003), Nordahl (2008), and summarized by Bauer et al. (2010)). The LSMC approach aims to find an optimal exercise point using only accessible information. For different points in time, the method compares between two values: the exercise and continuous values. The exercise value is the value if the option is exercised, while the continuous value is the value if the policyholder does not exercise the option and the contract continues without change. Following this strategy, policyholders exercise an option if its exercise value is larger than the continuous value. The original strategy determines the exercise value as a defined and deterministic cash flow when exercising the option. The continuous value is the PV of the future cash flows if the options are not exercised immediately. However, except for the surrender option, exercising an option does not always cause a defined immediate cash flow. We therefore make an adjustment to the original approach and define both the exercise and continuous values as the PV of future cash flows. For the special case of surrender options, we compare both the adjusted LSMC and the original method proposed by Longstaff and Schwartz (2001) in the Appendix. For our numerical example, the difference between the results of these two methods is negligible.

The original algorithm contains two approximations to converge the maximal option value (cf. Clément, Lamberton, and Protter (2002)). First, the continuous value at t denoted by $_{t}C(\vartheta)$ is approximated by the conditional combination of finite functions. The second approximation determines the value function via a least squares regression. We add a further two approximations for the exercise value, ϑ_t, the option value when the option is immediately exercised at t. Note that the option values are discounted for convenient comparison back to the beginning of the contract.

${}_tC(\vartheta)$ is approximated by ${}_tC(\vartheta) = E^{\mathbb{Q}}[{}_tC(\vartheta)|\mathcal{F}_t] \cong f(x_t^1...x_t^J)$, the conditional expected value under the risk-neutral distribution at year t. $x_t^1...x_t^J$ are J relevant variables (as all the possible information accessible at t). In our model, x^j with $j = 1...3$ includes the interest rate, r_t, the investment rate of return, S_t/S_{t-1}, and the adjusted benefit, γ_t at t.

The first approximation is given by:

$$
{}_tC(\vartheta) = E^{\mathbb{Q}}[{}_tC(\vartheta)|\mathcal{F}_t] \cong f(x_t^1...x_t^J) \tag{23}
$$

The second approximation includes K sets of basis functions to approximate $f(x_t^1...x_t^J)$ with α^k as constant coefficients. In our model, υ^k is a set of Laguerre polynomials. We set $K = 4$.[2]

$$
{}_tC(\vartheta) \cong f(x_t^1...x_t^J) \cong \sum_{k=0}^{K} \alpha_t^k \upsilon^k(x_t^1...x_t^J) \tag{24}
$$

The coefficients α^k are unknown so far. Using Monte Carlo simulation with $n = 1...N$ paths, we estimate α_t^k via least squares linear regression. In Longstaff and Schwartz (2001), these estimators are based solely on in-the-money paths to reduce computation effort. However, in our case, all paths should be considered since we are not focusing on standard put options (cf. Andreatta and Corradin (2003)). The estimator for α_t^k is provided by

$$
\hat{\alpha}_t^k = arg\min\{\sum_{n=1}^{N}[{}_t^nC(\vartheta) - \sum_{k=0}^{K} \alpha_t^k \upsilon^k({}^nx_t^1...{}^nx_t^J]\} \tag{25}
$$

with $\hat{\alpha}_t^k$,

$$
{}_t^n\hat{C}(\vartheta) = \sum_{k=0}^{K} \hat{\alpha}_t^k \upsilon^k({}^nx_t^1...{}^nx_t^J)
$$

As explained above, the option value at the time when the option is exercised is not known until maturity ($t = T$). Therefore, we introduce other approximations to calculate ϑ_t via $E^{\mathbb{Q}}[\vartheta_t|\mathcal{F}_t]$, the conditional expected discounted option value for a $\mathbb{Q}$ measure exercised immediately at t:

[2]When taking different values for K, our numerical results stabilize after $K = 4$.

$$
\begin{aligned}
\vartheta_t =& E^{\mathbb{Q}}[\vartheta_t|\mathcal{F}_t] \cong s(x_t^1...x_t^J) \cong \sum_{k=0}^{K} \alpha_t'^k \upsilon^k(x_t^1...x_t^J) \\
\hat{\alpha}_t'^k =& arg\min\{\sum_{n=1}^{N}[^n\vartheta_t - \sum_{k=0}^{K} \alpha_t'^k \upsilon^k(^nx_t^1...^nx_t^J]\} \\
^n\hat{\vartheta}_t =& \sum_{k=0}^{K} \hat{\alpha}_t'^k \upsilon^k(^nx_t^1...^nx_t^J)
\end{aligned}
\tag{26}
$$

Single Premium Payment Option Case (Paid-up Option Only / Surrender Option Only)

We aim to find an optimal exercise point, $^nt^*$, that maximizes the option value in each path, n, by using accessible information. At the end of each year, policyholders decide whether to exercise the option or not. The option should be exercised if the exercise value exceeds the continuous value.

The simulation procedure can be formally described as follows. For the Monte Carlo path, n, with $n = 1...N$:

1. At T, assume all $^nt^* = T$. The option value is zero as the contract matures without exercising the option. The optimal option value is given by $^n\vartheta_{^nt^*} = {}^n\vartheta_T = 0$.

2. One year before (at $T-1$), the continuous value is set at zero:

$$
{}_{T-1}^{n}C(\vartheta) = {}^n\vartheta_{^nt^*} = \vartheta_T = 0
$$

Policyholders decide at $t = T-1$ whether to exercise the option. If $^n\vartheta_{T-1}$ is positive (and hence exceeds the continuous value, which is zero) the option should be exercised and the optimal exercise point becomes $^nt^* = T-1$. Otherwise, the contract continues and $^nt^* = T$. In formal terms, we have:

$$
{}^nt^* = T-1, \text{ if } {}^n\vartheta_{T-1} > {}_{T-1}^{n}C(\vartheta) = 0. \text{ Otherwise, } {}^nt^* \text{ remains } T.
$$

Based on equation (26), we approximate $^n\vartheta_{T-1}$ by $^n\hat{\vartheta}_{T-1}$:

$$
{}^n\hat{\vartheta}_{T-1} = \sum_{k=0}^{K} \hat{\alpha}_{T-2}'^k \upsilon^k(^nx_{T-1}^1...^nx_{T-1}^J)
$$

3. With the help of equation (24) and (25), we find $\hat{\alpha}^k$ at $T-2$ to estimate ${}_t^nC(\vartheta)$.

$$
\begin{aligned}
{}^{n}_{T-2}C(\vartheta) &= {}^{n}\vartheta_{{}^{n}t^{*}} \\
\hat{\alpha}^{k}_{T-2} &= arg\min\{\sum_{n=1}^{N}[{}^{n}_{T-2}C(\vartheta) - \sum_{k=0}^{K}\alpha^{k}_{T-2}\upsilon^{k}({}^{n}x^{1}_{t}...{}^{n}x^{J}_{t}]\} \\
{}^{n}_{T-2}\hat{C}(\vartheta) &= \sum_{k=0}^{K}\hat{\alpha}^{k}_{T-2}\upsilon^{k}({}^{n}x^{1}_{t}...{}^{n}x^{J}_{t})
\end{aligned}
\tag{27}
$$

Again for $T-2$, if ${}^{n}\hat{\vartheta}_{T-2} > {}^{n}_{T-2}\hat{C}(\vartheta)$, ${}^{n}t^{*} = T-2$. Otherwise, ${}^{n}t^{*}$ remains unchanged.

Note that for equation (27), ${}^{n}_{T-2}C(\vartheta) = {}^{n}\vartheta_{{}^{n}t^{*}}$ instead of ${}^{n}\hat{\vartheta}_{{}^{n}t^{*}}$ since the latter may lead to biases when calculating the option value (cf. Longstaff and Schwartz (2001)).

4. With the same approach under the backwards algorithm, for $t = T-3...1$, we have:

(1) ${}^{n}_{t}C(\vartheta) = \vartheta_{{}^{n}t^{*}}$

(2) ${}^{n}_{t}\hat{C}(\vartheta) = \sum_{k=0}^{K}\hat{\alpha}^{k}_{t}\upsilon^{k}({}^{n}x^{1}_{t}...{}^{n}x^{J}_{t})$,

with $\hat{\alpha}^{k}_{t} = arg\min\{\sum_{n=1}^{N}[{}^{n}_{t}C(\vartheta) - \sum_{k=0}^{K}\alpha^{k}_{t}\upsilon^{k}({}^{n}x^{1}_{t}...{}^{n}x^{J}_{t})]\}$

and ${}^{n}_{t}\hat{\vartheta} = \sum_{k=0}^{K}\hat{\alpha}'^{k}_{t-1}\upsilon^{k}({}^{n}x^{1}_{t}...{}^{n}x^{J}_{t})$,

with $\hat{\alpha}'^{k}_{t} = arg\min\{\sum_{n=1}^{N}[{}^{n}_{t}\vartheta - \sum_{k=0}^{K}\alpha'^{k}_{t}\upsilon^{k}({}^{n}x^{1}_{t}...{}^{n}x^{J}_{t})]\}$

(3) If ${}^{n}\hat{\vartheta}_{t} >^{n}_{t} \hat{C}(\vartheta)$, ${}^{n}t^{*} = t$. Otherwise, ${}^{n}t^{*}$ remains unchanged.

5. With the algorithm above, the optimal option value equals the average of each path option value exercised at its respective point, ${}^{n}t^{*}$:

$$
{}^{LSMC}\vartheta = \frac{1}{N}\sum_{n=1}^{N}({}^{n}\vartheta_{{}^{n}t^{*}})
$$

Double Premium Payment Option Case

The double premium payment option case includes the combined paid-up and resumption option and combined paid-up and surrender option. We begin with the first option by using a single option exercise strategy described in the previous section. Conditional on the optimal exercise point for the first option, we add the second option. This method ensures the double payment option never has less value than the individual option.

Combined Paid-up and Resumption Option

The combined paid-up and resumption option, $\vartheta^{\mathcal{PR}}_{t,s}$ is a double option with two exercise points, t and s, where $s > t$. With the single-option method, we first determine the optimal exercise point, ${}^{n}t^{*}$, that maximizes the paid-up option value, ${}^{n}\vartheta^{\mathcal{P}}_{{}^{n}t^{*}}$, for the path n. If the optimal exercise point is T, the optimal strategy is not to exercise the paid-up option. In this case, we set ${}^{n}t^{*} = 0$.

Second, we find the resumption exercise point, ${}^{n}s^{*}$, to maximize the resumption option value. This option value, exercised at s, is given by $\vartheta^{\mathcal{R}}_{t^{*},s} = \vartheta^{\mathcal{PR}}_{t^{*},s} - \vartheta^{\mathcal{P}}_{t^{*}}$.

For the resumption exercise point, ${}^{n}s^{*}$, we have:

1. For ${}^{n}t* = 0$, the combination option has expired and has a value of 0. In this case, we set ${}^{n}s^{*} = T$.

2. For ${}^{n}t^{*} > 0$, the paid-up option has been exercised before maturity. We set ${}^{n}s^{*} = T$ as payment will not resume until maturity. Hence, in this case the resumption option has a value of zero:

$$ {}^{n}\vartheta^{\mathcal{R}}_{t^{*},T} = {}^{n}\vartheta^{\mathcal{PR}}_{{}^{n}t^{*},T} - {}^{n}\vartheta^{\mathcal{P}}_{{}^{n}t^{*}} = 0 $$

3. At $T-1$ and for $T-1 >^{n} t^{*} > 0$, the paid-up option has been exercised. Policyholders decide whether to resume the payment at $s = T-1$. The resumption option is exercised if the exercise value is positive (note that the continuous value ${}^{n}_{T-1}C(\vartheta) = {}^{n}\vartheta^{\mathcal{R}}_{{}^{n}t^{*},T}$ is zero).

${}^{n}\hat{\vartheta}^{\mathcal{R}}_{t^{*},T-1}$ is an estimator for the exercise value, ${}^{n}\vartheta^{\mathcal{R}}_{t^{*},T-1}$, at $T-1$ based on the least squares linear regression presented in equation (26).

If ${}^{n}\hat{\vartheta}^{\mathcal{R}}_{{}^{n}t^{*},T-1} > {}^{n}_{T-1}C(\vartheta) = 0$, then we set ${}^{n}s^{*} = T-1$; otherwise, ${}^{n}s^{*}$ remains the same.

4. With the same approach and using the backwards algorithm, for $s = T-2...2$ we set ${}^{n}s^{*} = s$ if ${}^{n}\hat{\vartheta}^{\mathcal{R}}_{t^{*},s} >^{n}_{s} \hat{C}(\vartheta)$ and ${}^{n}t^{*} < s$.

$\hat{\vartheta}^{\mathcal{R}}_{t^{*},s}$ and ${}^{n}_{s}\hat{C}(\vartheta)$ are two estimates for $\vartheta^{\mathcal{R}}_{t^{*},s}$ and ${}^{n}_{s}C(\vartheta) = {}^{n}\vartheta^{\mathcal{R}}_{{}^{n}t^{*},{}^{n}s^{*}}$ based on equations (24) to (26).

5. The optimal value of this combined paid-up and resumption option can be derived as follows:

$$ {}^{LSMC}\vartheta^{\mathcal{PR}} = \frac{1}{N}\sum_{n=1}^{N}({}^{n}\vartheta^{\mathcal{PR}}_{{}^{n}t^{*},{}^{n}s^{*}}) $$

Combined Paid-up and Surrender Option

The main difference between the combined paid-up and resumption option and the combined paid-up and surrender option is that the resumption option can only be exercised if the paid-up option is exercised first, i.e., $\vartheta^{\mathcal{PR}}_{t,s}$, $t < s$. However, policyholders can exercise the surrender option independently, even if the paid-up option has not yet been exercised. In formal terms, we have:

$$ \vartheta^{\mathcal{PS}}_{t,s} = \vartheta^{\mathcal{S}}_{s} \cdot \mathbb{1}_{t=0} + (\vartheta^{\mathcal{P}}_{t} + {\vartheta^{\mathcal{S}}}'_{t,s}) \cdot \mathbb{1}_{T \geq s > t > 0} \tag{28} $$

with ${\vartheta^{\mathcal{S}}}'_{t,s} = \vartheta^{\mathcal{PS}}_{t,s} - \vartheta^{\mathcal{P}}_t$ for $T \geq s > t > 0$.

First, the combined paid-up and surrender option is treated as two independent options. We begin by finding the optimal exercise point for one of these two options:

1. With ${}^n t^* = 0$ and ${}^n s^* = T$, both the paid-up and surrender option values are zero.

2. At $t = T-1$, the continuous value equals zero: ${}^n_{T-1}C(\vartheta) = 0$.

The option value is ${}^n\hat{\vartheta}^{\mathcal{PS}}_{{}^n t^*, {}^n s^*} = \max({}^n_{T-1}C(\vartheta), {}^n\hat{\vartheta}^{\mathcal{P}}_{T-1}, {}^n\hat{\vartheta}^{\mathcal{S}}_{T-1})$. ${}^n\hat{\vartheta}^{\mathcal{P}}_t$ and ${}^n\hat{\vartheta}^{\mathcal{S}}_s$ are two estimators of ${}^n\vartheta^{\mathcal{P}}_t$ and ${}^n\vartheta^{\mathcal{S}}_s$.

We consider three scenarios:

(1) ${}^n t^* = T-1$, ${}^n s^* = T$ if $\max({}^n_{T-1}C(\vartheta), {}^n\hat{\vartheta}^{\mathcal{P}}_{T-1}, {}^n\hat{\vartheta}^{\mathcal{S}}_{T-1}) = {}^n\hat{\vartheta}^{\mathcal{P}}_{T-1}$. The best strategy is to exercise the paid-up option.

(2) ${}^n t^* = 0$, ${}^n s^* = T-1$ if $\max({}^n_{T-1}C(\vartheta), {}^n\hat{\vartheta}^{\mathcal{P}}_{T-1}, {}^n\hat{\vartheta}^{\mathcal{S}}_{T-1}) = {}^n\hat{\vartheta}^{\mathcal{S}}_{T-1}$. Hence, the best strategy is to exercise the surrender option.

(3) ${}^n t^* = T$, ${}^n s^* = T$ if $\max({}^n_{T-1}C(\vartheta), {}^n\hat{\vartheta}^{\mathcal{P}}_{T-1}, {}^n\hat{\vartheta}^{\mathcal{S}}_{T-1}) = {}^n_{T-1}C(\vartheta) = 0$. In this case, policyholders should keep the two options, and the combined option value thus equals the continuous value.

3. With the backward algorithm and $t = T-2...1$, we have:

(1) ${}^n_t C(\vartheta) = {}^n\vartheta_{{}^n t^*, {}^n s^*}$

(2) ${}^n t^* = t$, ${}^n s^* = T$, if $\max({}^n\hat{\vartheta}^{\mathcal{P}}_t, {}^n\hat{\vartheta}^{\mathcal{S}}_t, {}^n_t\hat{C}(\vartheta)) = {}^n\hat{\vartheta}^{\mathcal{P}}_t$.

(3) ${}^n t^* = 0$, ${}^n s^* = t$, if $\max({}^n\hat{\vartheta}^{\mathcal{P}}_t, {}^n\hat{\vartheta}^{\mathcal{S}}_t, {}^n_t\hat{C}(\vartheta)) = {}^n\hat{\vartheta}^{\mathcal{S}}_t$.

(4) ${}^n t^*$, ${}^n s^*$ remain the same, if $\max({}^n\hat{\vartheta}^{\mathcal{P}}_t, {}^n\hat{\vartheta}^{\mathcal{S}}_t, {}^n_t\hat{C}(\vartheta)) = {}^n_t\hat{C}(\vartheta)$.

${}^n\hat{\vartheta}^{\mathcal{P}}_t, {}^n\hat{\vartheta}^{\mathcal{S}}_t, {}^n_t\hat{C}(\vartheta)$ are estimators for ${}^n\vartheta^{\mathcal{P}}_t, {}^n\vartheta^{\mathcal{S}}_t, {}^n_t C(\vartheta)$.

4. ${}^n t^* = 0$ means the best strategy is either to exercise the surrender option only (${}^n s^* < T$) or to hold the original contract until maturity (${}^n s^* = T$). If ${}^n t^* > 0$, the optimal strategy is to exercise the paid-up option at ${}^n t^*$. Given ${}^n t^* > 0$, we then derive the second optimal exercise point, ${}^n s^*$, to maximize the surrender option value, ${}^n{\vartheta^{\mathcal{S}}}'_{{}^n t^*, {}^n s} = {}^n\vartheta^{\mathcal{PS}}_{{}^n t^*, {}^n s} - {}^n\vartheta^{\mathcal{P}}_{{}^n t^*}$. The method is similar to that used for the combined paid-up and resumption option, and has the following three steps:

(1) For ${}^n t^* > 0$, given that the first option has been exercised, set ${}^n s^* = T$, i.e., the surrender option value is set at zero.

(2) At $T-1$, the continuous conditional value equals zero:

$$ {}^n_{T-1}C(\vartheta) = {}^n{\vartheta^{\mathcal{S}}}'_{{}^n t^*, {}^n s^*} = {}^n\vartheta^{\mathcal{PS}}_{t^*, T} - {}^n\vartheta^{\mathcal{P}}_{t^*} = 0 $$

If the first option has already been exercised ($T-1 >^n t^* > 0$), the policyholder decides whether to exercise the second option at ${}^n s^* = T-1$. The second option should be exercised if its value is positive. In formal terms, we have:

If ${}^n\hat{\vartheta}^{\mathcal{S}\prime}{}_{{}^n t^*,T-1} >^n_{T-1} C(\vartheta) = 0$, ${}^n s^* = T-1$.

(3) Based on the same approach using the backward algorithm for $s = T-2...2$, we approximate ${}^n\hat{\vartheta}^{\mathcal{S}\prime}{}_{{}^n t^*,s}$ and ${}^n_t\hat{C}(\vartheta)$ for ${}^n\vartheta^{\mathcal{S}\prime}{}_{{}^n t^*,s}$ and ${}^n_t C(\vartheta) = {}^n\vartheta^{\mathcal{S}\prime}{}_{{}^n t^*,{}^n s^*}$. For ${}^n t^* < s$, if ${}^n\hat{\vartheta}^{\mathcal{S}\prime}{}_{{}^n t^*,s} >^n_t \hat{C}(\vartheta)$, ${}^n s^* = s$. Otherwise, ${}^n s^*$ remains unchanged.

The PV of the simulated combined paid-up and surrender option is described as:

$$^{LSMC}\vartheta^{\mathcal{PS}} = \frac{1}{N}\sum_{n=1}^{N}({}^n\vartheta^{\mathcal{PS}}_{{}^n t^*,{}^n s^*})$$

with ${}^n\vartheta^{\mathcal{PR}}_{{}^n t^*,{}^n s^*}$ determined by equation (28).

4 Numerical Results

This section presents key results of our numerical analysis for discussion. In particular, we focus on the influence of the interest rate volatility, σ_I. Unless stated otherwise, the numerical results are gathered using a Monte Carlo simulation with $N = 10^4$. In the Appendix, we show that the results stabilize when N reaches 10^4.

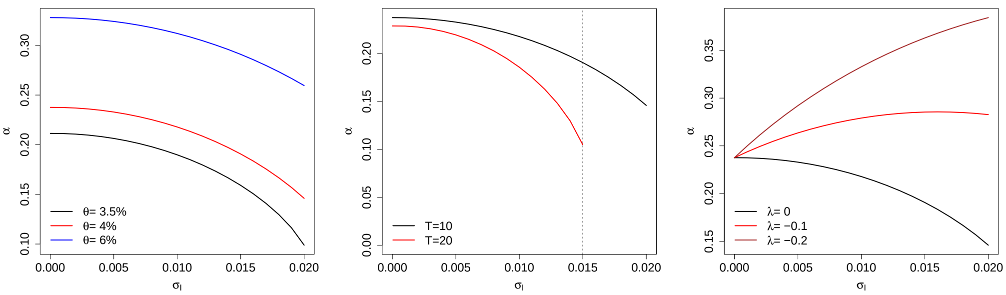
4.1 Basic Contract

We consider a basic contract with the following parameters: A 30-year-old female policyholder enters into a participating 10-year life insurance contract.[3] The premium per annum is 1,200 currency units and the yearly interest rate guaranteed is set to 3%. The investment return rate, $S_t/S_{t-1} - 1$, combines both the spot interest and the asset process as laid down in equation (8) using the risk-neutral measure, $\mathbb{Q}$. The asset volatility is fixed to $\sigma_s = 0.2$. The correlation between the asset and interest rate risk is $\rho = 0.05$. Under the Vasicek model, to obtain r, we use the parameters $\kappa = 8\%$, $r_0 = 4\%$, $\theta = 4\%$, and $\lambda = 0$. Based on these assumptions and using equation (2), the death benefit is 14,089 currency units. Table 2 summarizes the initial dataset.

[3]Mortality probabilities are for a 30-year-old US woman in 1994 based on the data from HMD, the Human Mortality Database.

B	constant premium	1200
γ	death benefit	14089
x	initial age	30
T	time to maturity	10
κ	interest reversion speed	8%
σ_s	asset volatility	0.2
ρ	correlation	5%
g	guarantee rate	3%
r_0	initial interest rate	4%
θ	long term interest mean	4%
λ	market price of risk	0%

Table 2: Parameter table for the base case



(a) α and σ_i for different θ (b) α and σ_i for different T (c) α and σ_i for different λ

Figure 1: Relationship between participation rate (α) and interest rate volatility (σ_I) for different values of long-term interest mean (θ), time to maturity (T), and market price of risk (λ)

Figure 1 shows the relationship between the participation rate, α, and interest rate volatility, σ_I, under different conditions. The participation rate, α, is derived such that the contract is fair at $t = 0$ (cf. Equation (9)). Figure 1(a) shows a clear trend: the higher σ_I is, the lower α gets. When σ_I increases – for the same contract with the same interest rate guarantee – insurers face higher risk and thus must lower the participation rate to ensure a risk-adequate return for shareholders. In addition, the curves with different values of θ move in parallel. For all σ_I from 0 to 2%, the model supports the same conclusion as drawn in Schmeiser and Wagner (2011): higher interest rates lead to a higher policyholder participation rate, α.

In Figure 1(b), α decreases when the time to maturity of the contract, T, is set to 20 years. Insurers face higher risk as T increases. Thus, in order to achieve risk-adequate returns for shareholders, the participation rate, α, must be reduced. When σ_I increases to more than 1.5% for $T = 20$, there is no α with $0 \leq \alpha \leq 1$ that satisfies the fairness conditions introduced in equation (9).

When $\lambda = 0$, we assume a risk-neutral market and hence no risk shift occurs when moving from the empirical to the risk-neutral measure. Whenever $\lambda < 0$, market participants are assumed to be risk averse. In this case, policyholders demand a higher participation rate, α, and/or a higher investment guarantee, g, whenever σ_I increases. Figure 1(c) shows that when $\lambda < 0$, a larger α is required compared to when $\lambda = 0$.

4.2 Premium Payment Option Value

In the following, we discuss the results of the four different premium payment options (i.e., a paid-up option, a combined paid-up and resumption option, a surrender option, and a combined paid-up and surrender option). The main focus is the option value ϑ, the usage ratio, and the value/premium ratio (V/P).

An option's value, ϑ, is defined as the difference between the PV of the contract with and without the premium payment option (equations (13), (17), (19), and (21)). In a Monte Carlo simulation, for certain paths, n, the option value may be negative for the whole contract period. When calculating the upper limit or PV of perfect information, the best strategy for these paths is not to exercise the options at all. Therefore, the usage ratio with the upper limit perspective shows, for the entire simulation, how many paths have a positive option value during the contract period and hence for how many paths the premium payment option should be exercised before the contract matures. In the case of LSMC, the usage ratio show how many paths have an optimal exercise point ${}^nt^* < T$.

The value/premium ratio (V/P) compares the option value and PV of the premium payments, if the premium payment options are exercised. In formal terms, we have:

$$V/P = \frac{\sum_{n=1}^{N} {}^n\vartheta}{\sum_{n=1}^{N} {}^n\text{PV Premium}} \tag{29}$$

with

a) ${}^n\text{PV Premium} = B \sum_{t=0}^{{}^nt^*} {}_tp_x (1 + {}^nr_t)^{-t}$ for the paid-up option, the surrender option, and the combined paid-up and surrender option, for which the premium payment stops at ${}^nt^*$

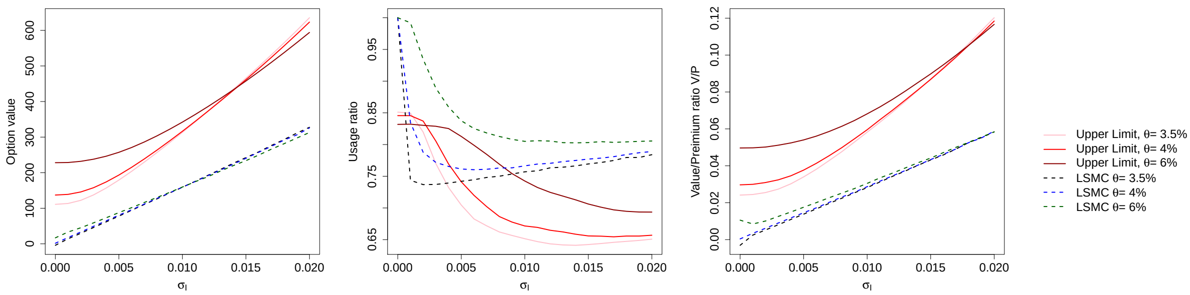
b) ${}^n\text{PV Premium} = B \sum_{t=0}^{{}^nt^*} {}_tp_x (1 + {}^nr_t)^{-t} + B \sum_{t={}^ns^*}^{T-1} {}_tp_x (1 + {}^nr_t)^{-t}$ for the combined paid-up and resumption option, for which the premium stops at ${}^nt^*$ but resumes at ${}^ns^*$

Paid-up Option

Figure 2 demonstrates the results for the basic contract with a paid-up option only. In Figure 2(a) with $\sigma_I = 0$ (deterministic term structure), the upper limit exhibits a similar structure to that presented in Schmeiser and Wagner (2011): A higher spot rate, θ, leads to a higher upper limit for the paid-up option. However, this interest-rate effect decreases as σ_I increases.

If $\sigma_I = 0$, LSMC may not be an efficient optimal strategy as its option values are close to zero for all three different θ. As σ_I increases, LSMC becomes a better approach for approximating an optimal exercise strategy. As σ_I increases, the value of the paid-up option under the LSMC also increases. However, the difference in value among various θ is small.

From Figure 2(c), as σ_I increases, V/P increases from 0% to 5.8%. The V/P ratio, based on the upper limit approach, increases even faster and reaches 11.8%.



(a) Option value (b) Usage ratio (c) Value/Premium ratio, V/P

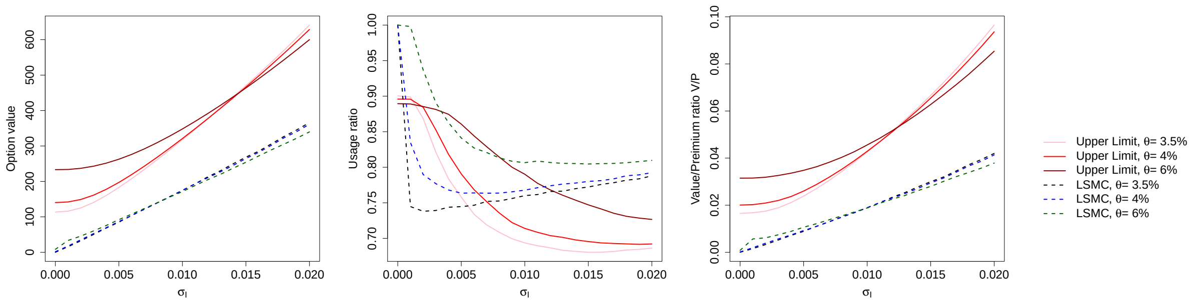
Figure 2: Paid-up option results for different σ_i

Combined Paid-up and Resumption Option

From Figure 3, we see a similar structure as that found for the paid-up option alone. The value of the combined paid-up and resumption option increases with increasing σ_I. The influence of changes in the interest rate, θ, on the upper limit decreases as σ_I increases.

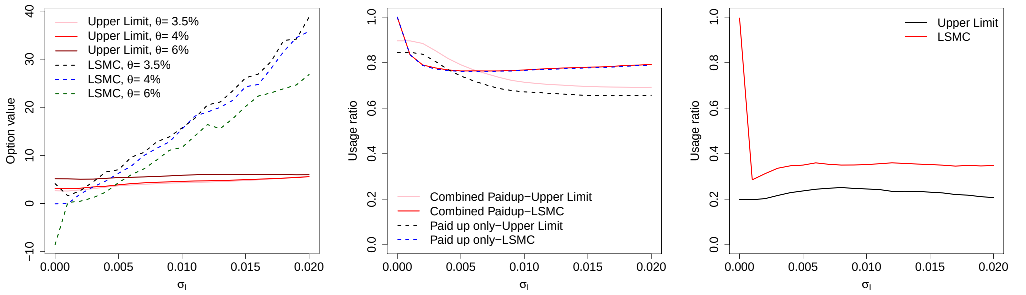
The resumption option in Figure 4(a) is derived as the difference between the combined paid-up and resumption option and the paid-up option alone. We find the resumption option for the upper limit has a lower value than the option value following the LSMC approach when σ_I is large. Using the LSMC method, the resumption option value increases while σ_I increases. Hence, with the resumption option, the option value derived using the optimal strategy under LSMC moves closer to the PV given perfect information. However, for both the LSMC and perfect-information cases, the V/P ratio of this combination option is lower than that for the paid-up option alone. When the resumption option is exercised, the premium payment resumes. As the extra resumption option has limited value compared to the resumed premium payments, the V/P ratio for this combined option actually decreases.

Figure 4(b) demonstrates and compares the usage ratio of the combined options. For the LSMC method, the usage ratio of the paid-up only option and that of the combined option are exactly the same as the LSMC method considering the paid-up option alone. From Figure 4(c), using the LSMC approach in cases where the paid-up option has been exercised, the resumption option can be used to adjust to future developments, which cannot be predicted at the time of exercising the paid-up option. For the upper limit case, policyholders use the resumption option less often as perfect information is assumed.



(a) Combined option value (b) Usage ratio (c) Value/Premium ratio, V/P

Figure 3: Combined paid-up and resumption option results for different σ_i

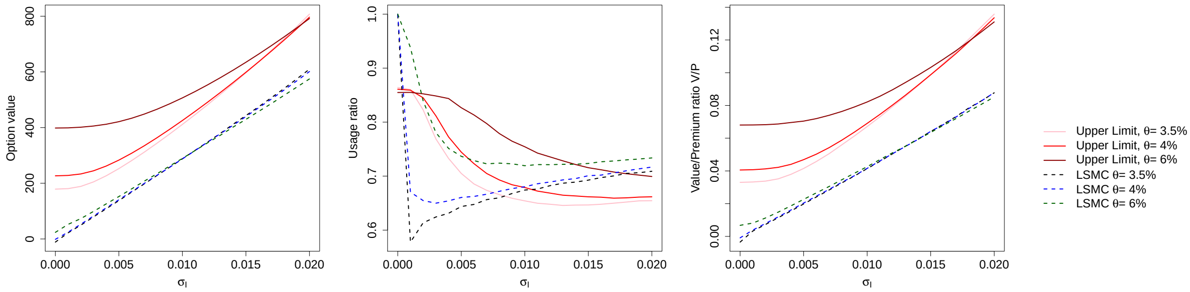


(a) Resumption option value comparison (b) Usage ratio for paid-up option comparison ($\theta = 4\%$) (c) Usage ratio for resumption option ($\theta = 4\%$)

Figure 4: Combined option comparison for different σ_i

Surrender Option

Like the paid-up options, the value of the surrender option increases with increasing σ_I. Moreover, surrender options are generally more valuable than paid-up options. If, e.g., $\sigma_I = 2\%$, the V/P ratio reaches 8.78% following the LSMC strategy and nearly 13.35% for the upper limit (for $\theta = 4\%$).



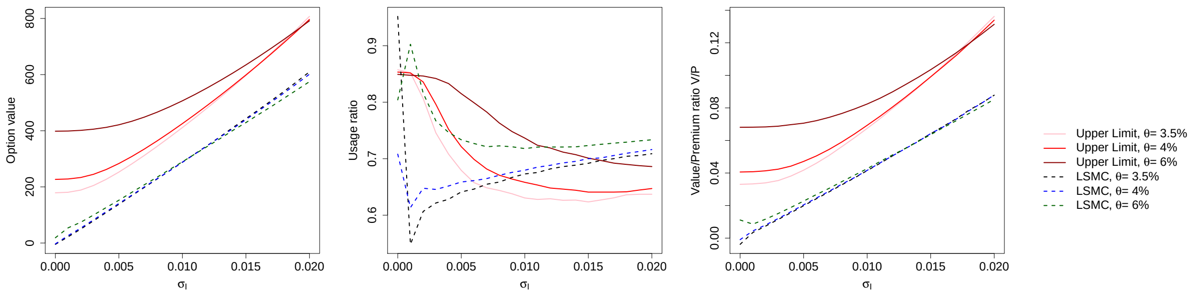
(a) Option value (b) Usage ratio (c) Value/Premium ratios, V/P

Figure 5: Surrender option results for different σ_i

Combined Paid-up and Surrender Option

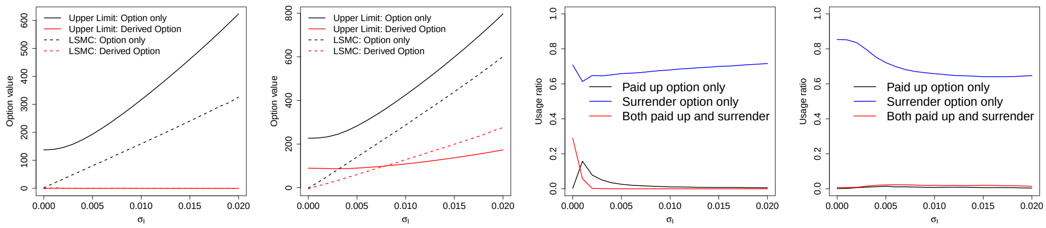
For the combined paid-up and surrender option, Figure 6 shows almost the same structure as in Figure 5 for the surrender option. Figure 7 compares the single premium payment option (paid-up only and surrender only) and the derived single option (the difference between the combined option and the paid-up option or the surrender option, respectively).

Figure 7(a) shows that the derived paid-up option value for both the LSMC and upper limit approaches is close to zero. Hence, when combining paid-up and surrender options, the value of the paid-up option becomes negligible. From Figure 7(c), it can be seen that more than 60% of the paths' best strategies involve exercising the surrender option alone. Using the LSMC approach, at the end of each year a decision is made about whether to exercise the option and which options to exercise based on the available information. The value of the surrender option is generally higher than that for the paid-up option. Therefore, a policyholder is more likely to exercise the surrender option. This strategy works as a reasonable optimal two-option strategy, as the upper limit approach also generates similar results: The paid-up option has negligible value (cf. Figure7(a)), and in most cases the surrender option is the only option used (cf. Figure 7(d)).



(a) Combined option value (b) Usage ratio (c) Value/Premium ratios, V/P

Figure 6: Combined paid-up and surrender option results for different σ_i



(a) Paid-up option value comparison (b) Surrender option value comparison (c) Usage ratios for LSMC method (d) Usage ratio for upper limit

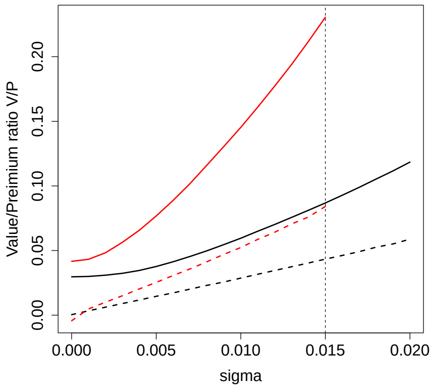
Figure 7: Combined option comparison for different σ_i ($\theta = 4\%$)

4.3 Sensitivity of Premium Payment Option

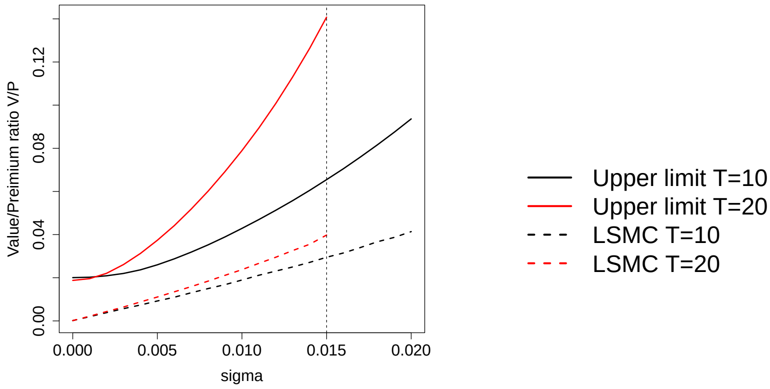
This section illustrates the influence of two other parameters on option value: contract duration, T, and the market price of risk (MPR).

The Influence of Contract Duration, T

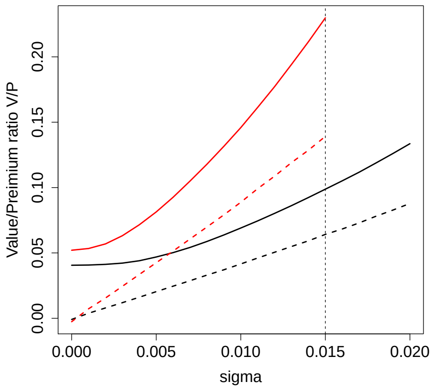
Figure 8 shows the V/P ratio of all four premium payment options for $T = 10$ and $T = 20$. For $T = 20$, no data is available for $\sigma_I \geq 1.5\%$ as there exists no α with $0 \leq \alpha \leq 1$ that satisfies the fair contract condition (cf. Figure 1(b)). V/P ratios increase dramatically when expanding the contract duration to $T = 20$.



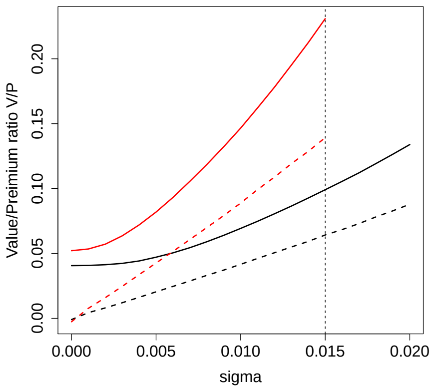
(a) V/P for paid-up option



(b) V/P for combined paid-up and resumption option



(c) V/P for surrender option

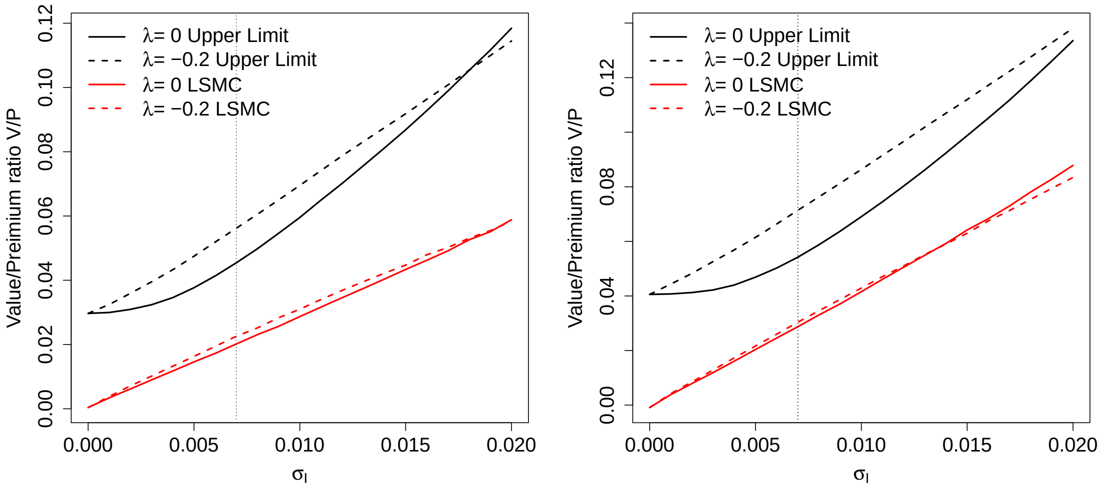


(d) V/P for combined paid-up and surrender option

Figure 8: Value/Premium ratio, V/P comparison between T=10 and T=20 for $\theta = 4\%$

The Influence of MPR, λ

From Figure 9, when varying λ, the participation rate α is always adjusted to satisfy the fair contract condition. The numerical results show that reducing the MPR slightly increases the value of the premium payment options.



(a) V/P for paid-up option

(b) V/P for surrender option

Figure 9: Value/Premium ratio, V/P with different λ for $\theta = 4\%$

5 Economic Interpretation and Outlook

The numerical results show that, when stochastic interest rates are taken into account, the fair values of premium payment options can be substantial. In addition, insurance companies face - in addition to pure random risk - extensive model and parameter risk in respect to the investigated options. Considering these factors, insurers may need to charge higher premiums than those proposed by the fair pricing concept shown in this paper.

Practitioners may argue that policyholders typically do not exercise premium payment options in a rational way (i.e., in the sense laid down in Chapter 3) and thus lower option prices based on observed exercise behavior could be sufficient. However, in such a case, insurance companies face some additional risk - policyholders could be advised about optimal exercise procedure and hence change their future behavior.

In most cases, insurance companies are not free to choose whether to offer premium payment options or not. For instance, a life insurance contract must have a surrender option by law in all insurance markets we are aware of. Under the assumptions taken in this paper, insurers must charge - in addition to the savings premium and premium for the term life part - substantial premiums for payment options to finance adequate risk management measures. This may reduce the attractiveness and hence the demand for life insurance contracts, given a competitive market with alternative products in the field of old-age provision. In addition, some policyholders may be convinced they will never use any of the premium options, resulting in no willingness to pay (even though such an assumption is irrational from an ex ante perspective).

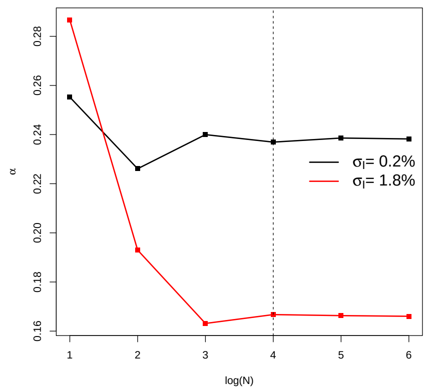
One way to tackle this problem from the insurer's point of view is to not base adjustment of benefits once an option is exercised on an ex ante fixed actuarial framework, but instead to pay out market values under any condition. The insurer would face no risk from premium payment options and need not charge any additional premium (because the option can never have positive value). On the other hand, the insurer would then be unable to promise policyholders a fixed payback under certain exercise procedures.

Alternatively, insurance companies could charge policyholders a fee whenever an option is exercised. The advantage here is that only those policyholders who exercise a premium payment option would need to pay. In this context, the premiums charged are lower, ceteris paribus, and may tempt customers to buy a life insurance contract. However, regulatory bodies are currently attempting to set minimum levels for surrender values in Europe to thwart such an approach and it could negatively influence the financial stability of life insurance companies if a large proportion of policyholders surrender their contracts at the same time (insurance run scenario).

A Appendix

A.1 Monte Carlo Convergence

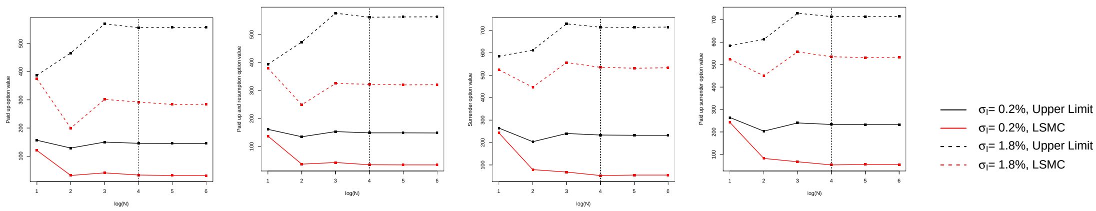
Figure 10 shows the speed of the convergence rate. We ran a Monte Carlo simulation for different N (from 10^1 to 10^6) with $\theta = 4\%$, $\sigma_I = 0.2\%$, and $\sigma_I = 1.8\%$. Both option values and the participation rate, α, stabilize when N reaches 10^4.



(a) α

N	$\sigma_I = 0.2\%$	$\sigma_I = 1.8\%$
10	0.25523	0.28663
100	0.22609	0.19299
1000	0.23998	0.16312
10000	0.23693	0.16673
100000	0.23859	0.16631
1000000	0.23822	0.16604

(b) α



(c) Paid-up value

(d) Paid-up resumption value

(e) Surrender value

(f) Paid-up surrender value

Figure 10: Convergence speed of α and the values of different premium payment options

A.2 Least-squares Monte Carlo Method (LSMC)

Figure 11 compares the LSMC strategy discussed in this paper and named "adjusted LSMC" to the Longstaff and Schwartz (2001) method. For our numerical example, we demonstrate that the results from the adjusted LSMC are slightly higher.

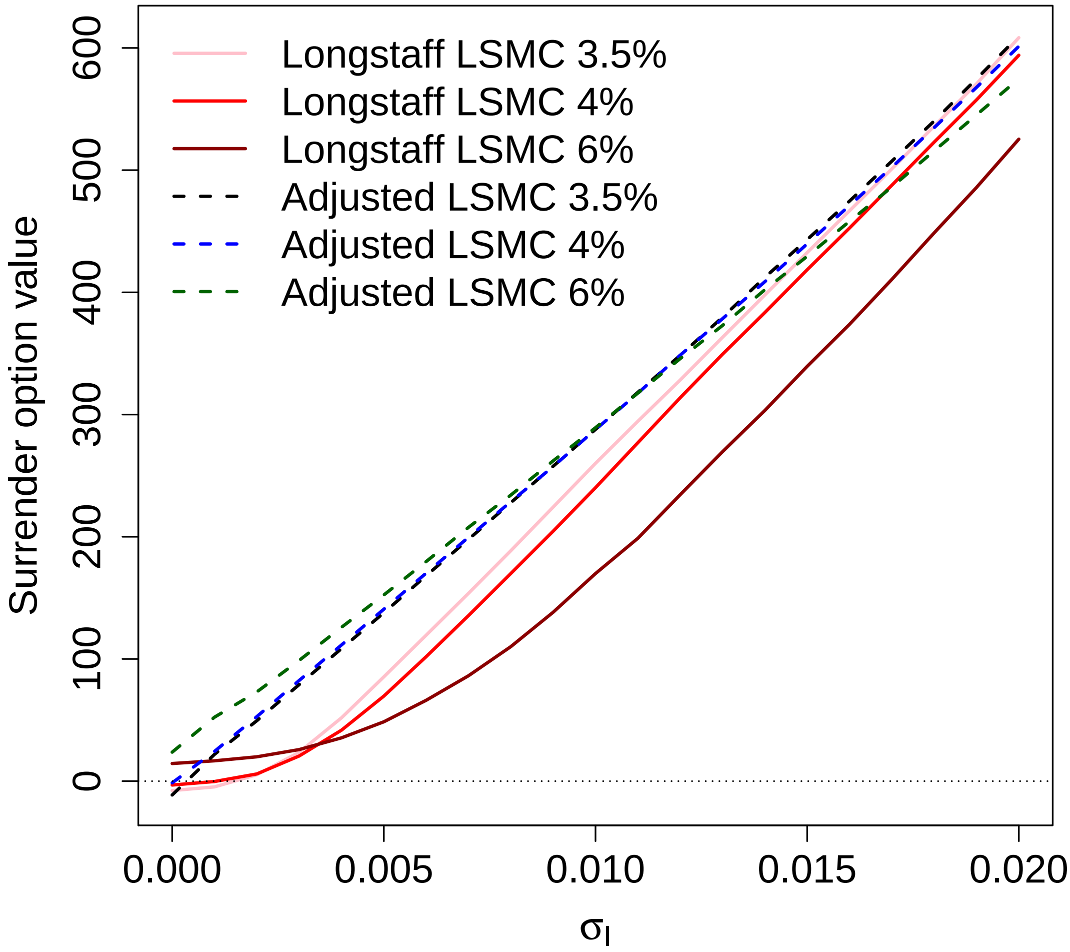


Figure 11: Comparison of surrender option value using different LSMC methods

To check the stability of LSMC approximation, we ran the first simulation and derived $\hat{\alpha}$ and $\hat{\alpha}'$ from equation (25) and (26). We then generated new simulation paths and determined their optimal exercise points using the derived $\hat{\alpha}$ and $\hat{\alpha}'$. The original result (the first simulation) and the second result (the new simulation) as out of sample (OoS) are compared in Table 3. We found no substantial differences, especially as σ_I increases.

Table 3: Out of sample check for paid-up option and surrender option values

	σ_I	Paid-up	$\mathcal{P}$-OoS	Diff	Surrender	$\mathcal{S}$-OoS	Diff
1	0.00%	1.65	0.74	75.50%	-2.18	-1.43	-41.33%
2	0.10%	17.73	17.12	3.51%	25.36	24.35	4.06%
3	0.20%	33.22	32.72	1.52%	54.28	52.80	2.76%
4	0.30%	49.22	47.83	2.86%	83.59	82.39	1.45%
5	0.40%	64.68	62.97	2.68%	112.60	111.44	1.04%
6	0.50%	80.60	78.56	2.57%	141.58	140.67	0.64%
7	0.60%	95.67	93.15	2.68%	170.79	170.20	0.35%
8	0.70%	111.83	109.29	2.30%	199.59	199.55	0.02%
9	0.80%	127.78	125.39	1.89%	229.09	228.70	0.17%
10	0.90%	143.37	141.18	1.54%	258.79	257.52	0.49%
11	1.00%	159.20	156.66	1.61%	287.76	287.70	0.02%
12	1.10%	175.07	172.22	1.64%	318.23	317.43	0.25%
13	1.20%	191.53	187.79	1.97%	348.17	348.55	0.11%
14	1.30%	207.31	204.13	1.54%	378.78	378.43	0.09%
15	1.40%	223.97	220.84	1.41%	409.23	408.73	0.12%
16	1.50%	240.46	236.73	1.57%	439.66	439.65	0.00%
17	1.60%	257.27	254.08	1.25%	471.43	471.09	0.07%
18	1.70%	273.95	270.29	1.34%	503.56	502.38	0.23%
19	1.80%	291.64	288.15	1.21%	535.32	534.78	0.10%
20	1.90%	306.16	305.61	0.18%	566.82	567.74	0.16%
21	2.00%	325.83	322.42	1.05%	600.45	601.34	0.15%

Reference

Andersen, L. B. (1999). A simple approach to the pricing of Bermudan swaptions in the multifactor libor market model. *Journal of Computational Finance*, *3*, 5–32.

Andreatta, G., & Corradin, S. (2003). Valuing the surrender options embedded in a portfolio of Italian life guaranteed participating policies: a least squares Monte Carlo approach. *Working Paper, University of California, Berkeley*.

Bacinello, A. R. (2003). Pricing guaranteed life insurance participating policies with periodical premiums and surrender option. *North American Actuarial Journal*, *7*(3), 1–17.

Bauer, D., Bergmann, D., & Kiesel, R. (2010). On the risk-neutral valuation of life insurance contracts with numerical methods in view. *ASTIN Bulletin*, *40*(1), 65–95.

Clément, E., Lamberton, D., & Protter, P. (2002). An analysis of a least squares regression method for American option pricing. *Finance and Stochastics*, *6*(4), 449–471.

Feodoria, M., & Förstemann, T. (2015). Lethal lapses: How a positive interest rate shock might stress German life insurers. *Bundesbank Discussion Paper*, *2015*(12).

Gatzert, N., & Schmeiser, H. (2008). Assessing the risk potential of premium payment options in participating life insurance contracts. *Journal of Risk and Insurance*, *75*(3), 691–712.

Kling, A., Russ, J., & Schmeiser, H. (2006). Analysis of embedded options in individual pension schemes in Germany. *Geneva Risk and Insurance review*, *31*(1), 43–60.

Kuo, W., Tsai, C., & Chen, W.-K. (2003). An empirical study on the lapse rate: the cointegration approach. *Journal of Risk & Insurance*, *70*(3), 489–508.

Linnemann, P. (2003). An actuarial analysis of participating life insurance. *Scandinavian Actuarial Journal*, *2003*(2), 153–176.

Longstaff, F. A., & Schwartz, E. (2001). Valuing American options by simulation: a simple least-squares approach. *Review of Financial Studies*, *14*(1), 113–147.

Nordahl, H. A. (2008). Valuation of life insurance surrender and exchange options. *Insurance: Mathematics and Economics*, *42*(3), 909–919.

Reuß, A., Ruß, J., & Wieland, J. (2016). Participating life insurance products with alternative guarantees: reconciling policyholders' and insurers' interests. *Risks*, *4*(2), 11.

Schmeiser, H., & Wagner, J. (2011). A joint valuation of premium payment and surrender options in participating life insurance contracts. *Insurance: Mathematics and Economics*, *49*(3), 580–596.

Vasicek, O. (1977). An equilibrium characterization of the terms structure. *Journal of Financial Economics*, *5*, 177–188.